Always Remember:

(i) If $f(-x) = -f(x)$, the function $f(x)$ is called an odd function.

(ii) If $f(-x) = f(x)$, the function $f(x)$ is called an even function.

(iii) $\sin n\pi = 0$ for any integer n. i.e. $n = 0, \pm 1, \pm 2, \ldots$

(iv) $\cos n\pi = -1 \; if \; n = \pm 1, \pm 3, \pm 5, \ldots$ (odd values)

(v) $\cos n\pi = 1 \; if \; n = 0, \pm 2, \pm 4, \pm 6 \; \ldots$ (even values)

(vi) $\int_{-\pi}^{\pi} \cos x \; dx = [\sin x]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0$

(vii) $\int_{-\pi}^{\pi} \sin x \; dx = [-\cos x]_{-\pi}^{\pi} = -[\cos(\pi) - \cos(-\pi)] = -[-1 - (-1)]$

(viii) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \; for \; m \neq n$

(ix) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \; for \; m \neq n$

(x) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \; for \; m, n = 1, 2, \ldots, \infty$

(xi) $\sin 2\theta = 2 \sin\theta \cos\theta$

(xii) $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

(xiii) $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

(xiv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(xv) $e^{i\theta} = \cos\theta + i \sin\theta$

(xvi) $e^{-i} = \cos\theta - i \sin\theta$

(xvii) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

(xviii) $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$Radian \rightarrow$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$Degree \rightarrow$	0°	30°	45°	60°	90°	180°
$\sin x$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1
$\tan x$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0

Periodic Function:

A periodic function is a function that repeats its values at regular intervals, for example, the trigonometric functions, which repeat at intervals of 2π radians.

If $f(x+p)=f(x)$ for every x then $f(x)$ is called periodic function and p is called period.

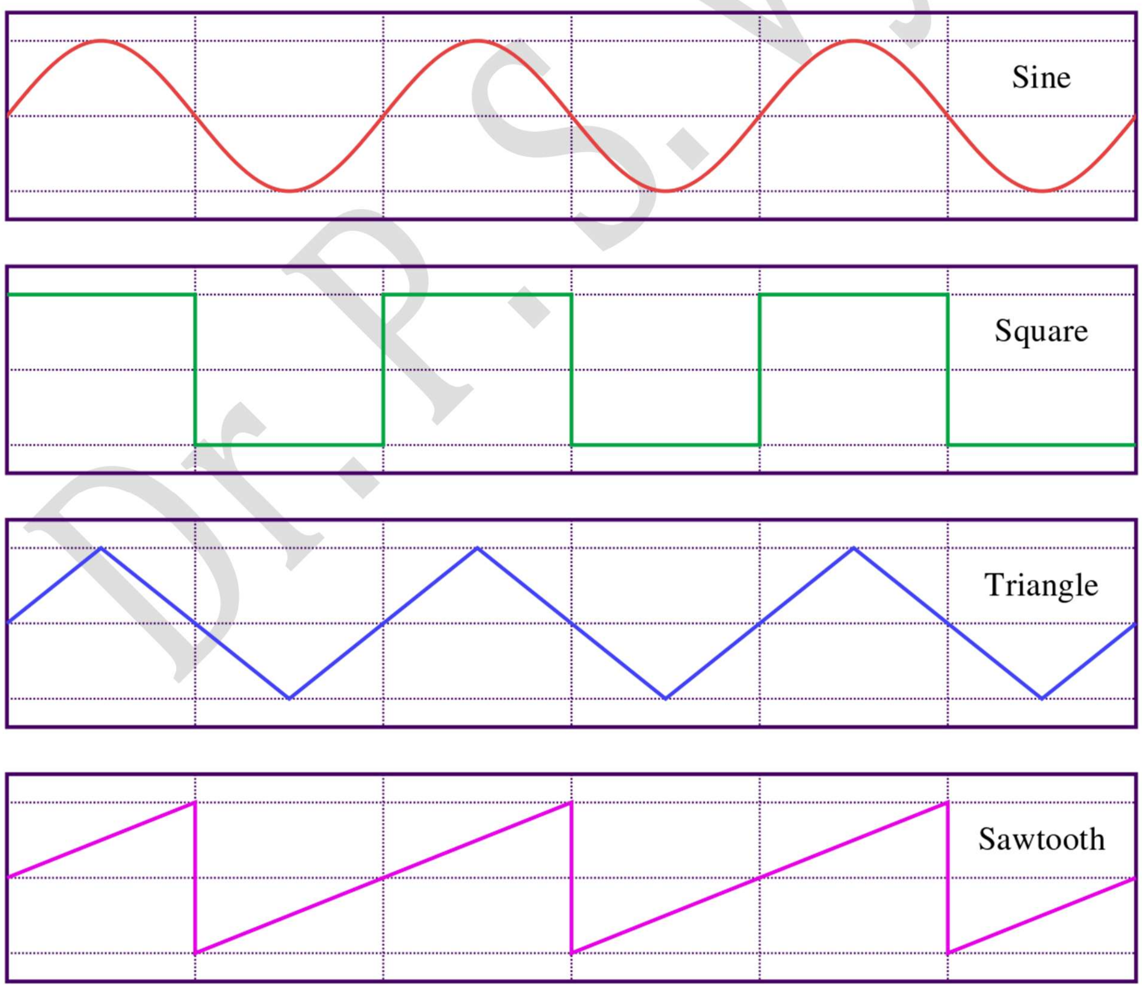
(i) If $f(x)=\sin x$ then
$f(x+2\pi)=\sin(x+2\pi)=\sin x=f(x)$
Therefore $\sin x$ is a periodic function with period 2π.

(ii) If $f(x)=\sin\left(\frac{2\pi x}{l}\right)$ then

$$f(x+l)=\sin\left(\frac{2\pi(x+l)}{l}\right)=\sin\left(\frac{2\pi x}{l}+\frac{2\pi l}{l}\right)=\sin\left(\frac{2\pi x}{l}+2\pi\right)=\sin\left(\frac{2\pi x}{l}\right)=f(x)$$

Therefore $\sin\left(\frac{2\pi x}{l}\right)$ is a periodic function with period l.

Different types of waves:



1. Definition and expansion of a function of Fourier series:

A Fourier series is a representation employed to express a periodic function $f(x)$ defined in an interval $(-\pi,\ \pi)$ a linear relation between the sines and cosines of the same period.

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

We want to determine values of the coefficients a_0, a_n and b_n.

(i) To calculate a_0:

To determine a_0, let us integrate both sides of equation (1) between limits $-\pi$ and π. We get

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + a_1 \int_{-\pi}^{\pi} \cos x\ dx + a_2 \int_{-\pi}^{\pi} \cos 2x\,dx + \cdots + a_n \int_{-\pi}^{\pi} \cos nx\,dx + \cdots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin x\,dx\ + b_2 \int_{-\pi}^{\pi} \sin 2x\,dx + \cdots + b_n \int_{-\pi}^{\pi} \sin nx\,dx + \cdots \qquad (2)$$

As we know that

$$\int_{-\pi}^{\pi} \cos nx\,dx = \int_{-\pi}^{\pi} \sin nx\,dx = 0\ for\ n = 1, 2, \dots, \infty \qquad (3)$$

In equation (2) RHS, except the first term all other terms become zero. Equation (2) becomes

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx = a_0[x]_{-\pi}^{\pi} = a_0[\pi - (-\pi)] = a_0[2\pi] = 2\pi a_0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\,d\vartheta \qquad (4)$$

by replacing variable of integration from x to ϑ to distinguish from $f(x)$.

(ii) To determine a_n:

Multiply both sides of equation (1) with $\cos nx$ and integrating between limits $-\pi$ and π, We get

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_0 \int_{-\pi}^{\pi} \cos nx\,dx + a_1 \int_{-\pi}^{\pi} \cos x \cos nx\ dx + a_2 \int_{-\pi}^{\pi} \cos 2x \cos nx\,dx$$
$$+ \cdots + a_n \int_{-\pi}^{\pi} \cos^2 nx\,dx + \cdots + b_1 \int_{-\pi}^{\pi} \sin x \cos nx\,dx\ + b_2 \int_{-\pi}^{\pi} \sin 2x \cos nx\,dx + \cdots$$
$$+ b_n \int_{-\pi}^{\pi} \sin nx \cos nx\,dx + \cdots \qquad (5)$$

As we know that

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \; for \; m \neq n \quad (6)$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \; for \; m, n = 1, 2, \dots, \infty \quad (7)$$

Therefore, all other terms vanish except only one term having $\cos^2 nx$, equation (5) becomes

$$\int_{-\pi}^{\pi} f(x) \cos nx \; dx = a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx$$

$$= \frac{a_n}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx \qquad \left(As \; \cos^2\theta = \frac{1 + \cos 2\theta}{2} \right)$$

$$= \frac{a_n}{2} \left[x + \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \frac{a_n}{2} \left[\pi - (-\pi) + \frac{\sin 2n\pi - \sin(-2n\pi)}{2n} \right]$$

$$= \frac{a_n}{2} \left[\pi + \pi + \frac{0 - 0}{2n} \right] = \frac{a_n}{2} [2\pi] = a_n \pi$$

$$\therefore \int_{-\pi}^{\pi} f(x) \cos nx \; dx = a_n \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\vartheta) \cos n\vartheta \; d\vartheta \qquad (8)$$

(iii) To determine b_n:

Multiply both sides of equation (1) with $\sin nx$ and integrating between limits $-\pi$ and π, We get

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = a_0 \int_{-\pi}^{\pi} \sin nx \, dx + a_1 \int_{-\pi}^{\pi} \cos x \sin nx \; dx + a_2 \int_{-\pi}^{\pi} \cos 2x \sin nx \, dx$$

$$+ \cdots + a_n \int_{-\pi}^{\pi} \cos nx \sin nx \, dx + \cdots + b_1 \int_{-\pi}^{\pi} \sin x \sin nx \, dx \; + b_2 \int_{-\pi}^{\pi} \sin 2x \sin nx \, dx + \cdots$$

$$+ b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx + \cdots \qquad (9)$$

Therefore, all other terms vanish except only one term having $\sin^2 nx$, equation (9) becomes

$$\int_{-\pi}^{\pi} f(x) \sin nx \; dx = b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

$$= \frac{b_n}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx \qquad \left(As \; \sin^2\theta = \frac{1 - \cos 2\theta}{2} \right)$$

$$= \frac{b_n}{2} \left[x - \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \frac{b_n}{2} \left[\pi - (-\pi) - \frac{\sin 2n\pi - \sin(-2n\pi)}{2n} \right]$$

$$= \frac{b_n}{2}\left[\pi + \pi - \frac{0-0}{2n}\right] = \frac{b_n}{2}[2\pi] = b_n\pi$$

$$\therefore \int_{-\pi}^{\pi} f(x)\sin nx \; dx = b_n\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta \qquad (10)$$

Substituting values of a_0, a_n & b_n in equation (1), we get

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta + \frac{1}{\pi}\sum_{n=1}^{\infty}\sin nx\int_{-\pi}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta \quad (11)$$

The expansion as shown in RHS of equation (11) is called Fourier series for $f(x)$, in the interval $-\pi \ll x \ll \pi$ and a_0, a_n & b_n are known as Fourier's constants for $f(x)$.

Equation (11) may be also written as

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} f(\vartheta)\cos n(x-\vartheta) \; d\vartheta\right] \quad (12)$$

$[As \;\; \cos(A-B) = \cos A\cos B + \sin A\sin B]$

Deduction from equation (11):

(i) If $f(x)$ be an odd function of x i.e. $f(-x) = -f(x)$, then first & second term of equation (11) become zero. i.e.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = 0, \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta = 0 \quad (13)$$

In third term of equation (11) we have

$$\int_{-\pi}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta = 2\int_{0}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta \quad (14)$$

Therefore equation (11) becomes:

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin nx\int_{0}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta \quad (15)$$

(ii) If $f(x)$ be an even function of x, i.e. $f(-x) = f(x)$ then

$$\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = 2\int_{0}^{\pi} f(\vartheta)\, d\vartheta \qquad (16)$$

$$\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta = 2\int_{0}^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta \quad (17)$$

$$\int_{-\pi}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta = 0 \quad (18)$$

$$\therefore \; f(x) = \frac{1}{\pi}\int_0^{\pi} f(\vartheta)\, d\vartheta + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx \int_0^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta \quad (19)$$

Corollary 1. To find a cosine series of $f(x)$ when $0 \ll x \ll \pi$.

Let us assume that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad (20)$$

Integrating both sides from 0 to π, equation (20) becomes

$$\int_0^{\pi} f(x)\, dx = a_0 \int_0^{\pi} dx = a_0[x]_0^{\pi} = a_0[\pi - 0] = a_0[\pi] = \pi a_0$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} f(\vartheta)\, d\vartheta \qquad (21)$$

Again, multiplying both sides of equation (20) by $\cos nx$ and integrating from 0 to π, we get

$$\int_0^{\pi} f(x)\cos nx\, dx = a_0\int_0^{\pi}\cos nx\, dx + a_1\int_0^{\pi}\cos x\cos nx \; dx + \cdots + a_n\int_0^{\pi}\cos^2 nx\, dx + \cdots (22)$$

Therefore, all other terms vanish except only one term having $\cos^2 nx$, equation (22) becomes

$$\int_0^{\pi} f(x)\cos nx \; dx = a_n \int_0^{\pi}\cos^2 nx\, dx$$

$$= \frac{a_n}{2}\int_0^{\pi}(1 + \cos 2nx)\, dx$$

$$= \frac{a_n}{2}\left[x + \frac{\sin 2nx}{2n}\right]_0^{\pi} = \frac{a_n}{2}\left[\pi - 0 + \frac{\sin 2n\pi - \sin(0)}{2n}\right] = \frac{a_n}{2}\left[\pi + \frac{0-0}{2n}\right] = \frac{a_n}{2}[\pi] = \frac{a_n\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \; dx = \frac{2}{\pi}\int_0^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta \qquad (23)$$

Hence,

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(\vartheta)\, d\vartheta + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\left[\int_0^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta\right] \quad (24)$$

Corollary 2. To find a sine series for $f(x)$, when $0 \ll x \ll \pi$.

Let us assume that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (25)$$

Multiply both sides of equation (25) with $\sin nx$ and integrating between limits 0 and π, We get

$$\int_0^\pi f(x) \sin nx \, dx = b_1 \int_0^\pi \sin x \sin nx \, dx + \cdots + b_n \int_0^\pi \sin^2 nx \, dx + \cdots (26)$$

All other terms vanish except only one term having $\sin^2 nx$, equation (9) becomes

$$\int_0^\pi f(x) \sin nx \, dx = b_n \int_0^\pi \sin^2 nx \, dx$$

$$= \frac{b_n}{2} \int_0^\pi (1 - \cos 2nx) \, dx \qquad \left(As \ \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \right)$$

$$= \frac{b_n}{2} \left[x - \frac{\sin 2nx}{2n} \right]_0^\pi = \frac{b_n}{2} \left[\pi - 0 - \frac{\sin 2n\pi - \sin(0)}{2n} \right] = \frac{b_n}{2} \left[\pi - \frac{0-0}{2n} \right] = \frac{b_n}{2} [\pi] = \frac{b_n \pi}{2}$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi f(\vartheta) \sin n\vartheta \, d\vartheta \qquad (27)$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \int_0^\pi f(\vartheta) \sin n\vartheta \, d\vartheta \qquad (28)$$

Corollary 3. To obtain Fourier series for function $f(x)$ in the interval $(-l, l)$ with period $2l$, replace x by $\frac{\pi x}{l}$, equation (1) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Or

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \qquad (29)$$

Where

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) \, dx , \qquad a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx , \qquad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \quad (30)$$

Complex Representation of a Fourier Series:

Consider a function $f(t)$ which is periodic with a period $\tau = \frac{2\pi}{\omega}$. then we can write

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t} \qquad (1)$$

$$\omega = \frac{2\pi}{\tau}, \qquad f(t+\tau) = f(t) \quad (2)$$

Here $f(t)$ being defined in $(-\infty,\ \infty)$. Now RHS of equation (1) being real, the coefficients of the series on the RHS of equation (1) must be such that no imaginary terms occur.

Integrating equation (1) over 0 to τ, we have

$$\int_0^\tau f(t)dt = \int_0^\tau \left[\sum_{n=-\infty}^{\infty} a_n e^{in\omega t}\right] dt \quad (3)$$

(In equation (3) summation is over number n and integration is on t, since both are independent of each other, we can change their order.)

$$\int_0^\tau f(t)dt = \sum_{n=-\infty}^{\infty} a_n \left[\int_0^\tau e^{in\omega t} dt\right] \quad (4)$$

To solve equation (4), we have to solve the integral separately.

We know that

$$\int_a^b e^{m\theta} d\theta = \left[\frac{e^{m\theta}}{m}\right]_a^b \ for\ \boldsymbol{m \neq 0} \quad (5)$$

(From equation (5) it is seen that for $m = 0$, the RHS becomes ∞. Therefore, we can use relation (5) only for $\boldsymbol{m \neq 0}$)

Therefore, we calculate $n = 0$ and $n \neq 0$ terms separately in solving the integral of equation (4).

(i) For $n = 0$

$$\int_0^\tau e^{in\omega t} dt = \int_0^\tau e^0 dt = \int_0^\tau dt = [t]_0^\tau = \tau - 0 = \tau$$

$$\boldsymbol{\int_0^\tau e^{in\omega t} dt = \tau\ for\ n = 0} \quad (6)$$

(ii) For $n \neq 0$

$$\int_0^\tau e^{in\omega t} dt = \left[\frac{e^{in\omega t}}{in\omega}\right]_0^\tau = \frac{1}{in\omega}\left[e^{in\omega t}\right]_0^\tau$$

$$= \frac{1}{in\omega}\left[e^{in\omega\tau} - e^0\right]$$

(From equation (2) $\omega = \frac{2\pi}{\tau}$ or $\omega\tau = 2\pi$)

$$= \frac{1}{in\omega}\left[e^{in2\pi} - 1\right] = \frac{1}{in\omega}\left[e^{i2n\pi} - 1\right]$$

$$= \frac{1}{in\omega}[\cos 2n\pi + i \sin 2n\pi - 1] = \frac{1}{in\omega}[1 + 0 - 1] = 0$$

(As $\sin 2n\pi = 0$, no imaginary terms appear in the equation)

$$\int_{\mathbf{0}}^{\boldsymbol{\tau}} \boldsymbol{e^{in\omega t}dt} = \mathbf{0} \;\; for\; \boldsymbol{n} \neq \mathbf{0} \quad (7)$$

by rewriting equation (6) and (7),

$$\int_{\mathbf{0}}^{\boldsymbol{\tau}} \boldsymbol{e^{in\omega t}dt} = \begin{matrix} \mathbf{0} \;\; for\; \boldsymbol{n} \neq \mathbf{0} \\ \boldsymbol{\tau}\; for\; \boldsymbol{n} = \mathbf{0} \end{matrix} \quad (8)$$

Therefore equation (4) becomes:

$$\int_0^{\tau} f(t)dt = a_0\tau$$

$$\therefore \; a_0 = \frac{1}{\tau}\int_0^{\tau} f(t)dt = \overline{f(t)} \quad (9)$$

Here $\overline{f(t)}$ denotes the mean value of $f(t)$.

Now multiply equation (1) by $e^{-in\omega t}$ and integrating over 0 to τ, we have

$$\int_0^{\tau} f(t)\, e^{-in\omega t}\, dt = \int_0^{\tau} \left[\sum_{n=-\infty}^{\infty} a_n e^{in\omega}\right] e^{-in} \quad dt$$

$$\int_0^{\tau} f(t)\, e^{-in\omega t}\, dt = \cdots + a_{-2}\int_0^{\tau} e^{i(-2)\omega t} e^{-in\omega t}\, dt + a_{-1}\int_0^{\tau} e^{i(-1)\omega t} e^{-in\omega t}\, dt$$

$$+a_0\int_0^{\tau} e^{i(0)\omega t} e^{-in\omega t}\, dt + a_1\int_0^{\tau} e^{i(1)\omega t} e^{-in\omega t}\, dt + a_2\int_0^{\tau} e^{i(2)\omega t} e^{-in} \quad dt + \cdots$$

$$+ a_n\int_0^{\tau} e^{i(n)\omega t} e^{-in\omega} \;\; dt + \cdots \quad (10)$$

From equation (7), only one term of coefficient a_n survives, all other terms vanish

$$\int_0^{\tau} f(t)\, e^{-in\omega t}\, dt = a_n\int_0^{\tau} dt = a_n\tau$$

$$a_n = \frac{1}{\tau}\int_0^{\tau} f(t)\, e^{-in\omega} \;\; dt \quad (11)$$

Replacing n by $-n$ in equation (11), we get

$$a_{-n} = \frac{1}{\tau}\int_0^{\tau} f(t)\, e^{in\omega t}\, dt \quad (12)$$

From equation (11) and (12), we conclude that

$$a_{-n} = \overline{a_n} \quad (13)$$

In order to find the usual real form of the Fourier series, equation (1) can be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t} = \sum_{n=-\infty}^{-1} a_n e^{in\omega t} + a_0 + \sum_{n=1}^{\infty} a_n e^{in\omega t}$$

$$= \sum_{n=-1}^{-\infty} a_n e^{in\omega t} + a_0 + \sum_{n=1}^{\infty} a_n e^{in\omega t}$$

$$= \sum_{n=1}^{\infty} a_{-n} e^{-in\omega} + a_0 + \sum_{n=1}^{\infty} a_n e^{in\omega t}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n e^{in\omega t} + a_{-n} e^{-in\omega t}\right] \quad (14)$$

We know that: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \; e^{in\omega t} = \cos n\omega t + i\sin n\omega t\,, \qquad e^{-in\omega t} = \cos n\omega t - i\sin n\omega t \quad (15)$$

Therefore equation (14) becomes

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n(\cos n\omega t + i\sin n\omega t) + a_{-n}(\cos n\omega t - i\sin n\omega t)]$$

By rearranging the terms, we have

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n + a_{-n})\cos n\omega t + \sum_{n=1}^{\infty} i(a_n - a_{-n})\sin n\omega t \quad (16)$$

If we take

$$(a_n + a_{-n}) = \alpha_n, \qquad i(a_n - a_{-n}) = \beta_n, \qquad 2a_n = \alpha_0 \quad (17)$$

Equation (16) becomes:

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\omega t + \sum_{n=1}^{\infty} \beta_n \sin n\omega t \quad (18)$$

Which is the same form of Fourier series.

We can determine the coefficients α_n and β_n by substituting values of equation (11) and (12) in equation (17).

$$\alpha_n = a_n + a_{-n} = \frac{1}{\tau}\int_0^\tau f(t)\left[e^{-in\omega t} + e^{in\omega t}\right] dt$$

$$= \frac{1}{\tau}\int_0^\tau f(t)\,[\cos n\omega t - i\sin n\omega t + \cos n\omega t + i\sin n\omega t]\, dt$$

$$\therefore\ \alpha_n = \frac{2}{\tau}\int_0^\tau f(t)\ \cos n\omega t\, dt \qquad (19)$$

$$\beta_n = i(a_n - a_{-n}) = \frac{1}{\tau}\int_0^\tau f(t)\, i\left[e^{-in\omega t} - e^{in\omega t}\right] dt$$

$$= \frac{1}{\tau}\int_0^\tau f(t)\, i\,[(\cos n\omega t - i\sin n\omega t) - (\cos n\omega t + i\sin n\omega t)]\, dt$$

$$= \frac{1}{\tau}\int_0^\tau f(t)\, i\,[-2i\sin n\omega t]\, dt$$

As $i^2 = -1,\ -i^2 = 1$

$$\beta_n = \frac{2}{\tau}\int_0^\tau f(t)\ \sin n\omega t\ dt \qquad (20)$$

Example:

Obtain Fourier series for the expansion of $f(x) = x\sin x$ in the interval $-\pi \ll x \ll \pi$. Hence deduce that $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$

Solution: the given function $f(x) = x\sin x$ is an even function of x in the interval $-\pi \ll x \ll \pi$. Hence the Fourier expansion of the given function $x\sin x$ would contain only cosine terms.

$$f(x) = x\sin x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{\pi}\int_0^{\pi} \vartheta \sin\vartheta \, d\vartheta \qquad (2)$$

We know the rule for integration by parts: for $u(x)$ & $v(x)$

$$\int_a^b u \cdot v \, dx = \left[u \int v \, dx - \int \left\{\frac{du}{dx} \int v \, dx\right\} dx\right]_a^b \qquad (3)$$

By taking $u = \vartheta, v = \sin v$

$$\int_0^{\pi} \vartheta \cdot \sin\vartheta \; d\vartheta = \left[\vartheta \int \sin\vartheta \; d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta} \int \sin\vartheta \; d\vartheta\right\} d\vartheta\right]_0^{\pi}$$

$$= \left[\vartheta\,(-\cos\vartheta) - \int \{-\cos\vartheta\} d\vartheta\right]_0^{\pi}$$

$$= -[\vartheta\cos\vartheta]_0^{\pi} + [\sin\vartheta]_0^{\pi}$$

$$= -[\pi\cos\pi - 0] + [\sin\pi - \sin 0] = -\pi\cos\pi = -\pi(-1) = \pi$$

$$\int_0^{\pi} \vartheta \cdot \sin\vartheta \; d\vartheta = \pi \qquad (4)$$

Therefore equation (2) becomes:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \vartheta \sin\vartheta \, d\vartheta = \frac{1}{\pi} \times \pi = 1 \quad \therefore \quad a_0 = 1 \qquad (5)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta = \frac{2}{\pi}\int_0^{\pi} \vartheta \sin\vartheta \cos n\vartheta \; d\vartheta \qquad (6)$$

We know that: $\mathbf{2\ \cos \boldsymbol{A} \sin \boldsymbol{B} = \sin(\boldsymbol{A}+\boldsymbol{B}) - \sin(\boldsymbol{A}-\boldsymbol{B})}$

$$\therefore \quad 2\cos n\vartheta \; \sin\vartheta = \sin(n+1)\vartheta - \sin(n-1)\vartheta \qquad (7)$$

Therefore equation (6) becomes

$$a_n = \frac{2}{\pi}\int_0^{\pi} \vartheta \sin\vartheta \cos n\vartheta \; d\vartheta = \frac{1}{\pi}\int_0^{\pi} \vartheta\,[\sin(n+1)\vartheta - \sin(n-1)\vartheta] d\vartheta$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} \vartheta \sin(n+1)\vartheta \; d\vartheta - \int_0^{\pi} \vartheta \sin(n-1)\vartheta \; d\vartheta\right] \qquad (8)$$

Now applying rule for integration by parts, we take $u = \vartheta, v = \sin(n+1)\vartheta$ in first integral of equation (8), we have

$$\int_0^\pi \vartheta \sin(n+1)\vartheta \; d\vartheta = \left[\vartheta \int \sin(n+1)\vartheta \; d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta} \int \sin(n+1)\vartheta \; d\vartheta\right\} d\vartheta\right]_0^\pi$$

$$= \left[-\vartheta \; \frac{\cos(n+1)\vartheta}{n+1} - \int \left\{-\frac{\cos(n+1)\vartheta}{n+1}\right\} d\vartheta\right]_0^\pi$$

$$= \left[-\vartheta \; \frac{\cos(n+1)\vartheta}{n+1} + \frac{\sin(n+1)\vartheta}{(n+1)^2}\right]_0^\pi$$

$$= -\pi \; \frac{\cos(n+1)\pi}{n+1} + 0 + \frac{\sin(n+1)\pi - \sin 0}{(n+1)^2}$$

$$\therefore \int_0^\pi \vartheta \sin(n+1)\vartheta \; d\vartheta = -\pi \; \frac{\cos(n+1)\pi}{n+1} \qquad (9)$$

Similarly,

$$\int_0^\pi \vartheta \sin(n-1)\vartheta \; d\vartheta = -\pi \; \frac{\cos(n-1)\pi}{n-1} \qquad (10)$$

Therefore equation (8) becomes

$$a_n = \frac{1}{\pi}\left[\int_0^\pi \vartheta \sin(n+1)\vartheta \; d\vartheta - \int_0^\pi \vartheta \sin(n-1)\vartheta \; d\vartheta\right]$$

$$= \frac{1}{\pi}\left[-\pi \; \frac{\cos(n+1)\pi}{n+1} - \left\{-\pi \; \frac{\cos(n-1)\pi}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\pi \; \frac{\cos(n+1)\pi}{n+1} + \pi \; \frac{\cos(n-1)\pi}{n-1}\right]$$

$$= -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}$$

As $\cos(\pi+\theta) = -\cos\theta$, $\cos(\pi-\theta) = -\cos\theta$

$$a_n = \frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} = \left[\frac{(n-1)-(n+1)}{(n+1)\,(n-1)}\right] \cos n\pi$$

$$a_n = \left[\frac{n-1-n-1}{n^2-1}\right] \cos n\pi$$

$$a_n = -\frac{2\cos n\pi}{n^2-1} \qquad (11)$$

If we take $n = 1$, above equation shows that denominator becomes zero and a_n becomes ∞. Therefore, above equation is true only for $n \neq 1$.

For $n = 1$ equation (6) becomes:

$$a_n = a_1 = \frac{2}{\pi}\int_0^{\pi} \vartheta \sin\vartheta \cos\vartheta \; d\vartheta = \frac{1}{\pi}\int_0^{\pi} \vartheta \,\{2\sin\vartheta\cos\vartheta\}\, d\vartheta$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \vartheta \sin 2\vartheta \; d\vartheta \qquad (12)$$

By applying rule for integration by parts, we take $u = \vartheta, v = \sin n\vartheta$

$$a_1 = \frac{1}{\pi}\left[\vartheta \int \sin 2\vartheta \; d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta}\int \sin 2\vartheta \; d\vartheta\right\} d\vartheta\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\vartheta\frac{\cos 2\vartheta}{2} - \int\left\{-\frac{\cos 2\vartheta}{2}\right\} d\vartheta\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\vartheta\frac{\cos 2\vartheta}{2} + \frac{\sin 2\vartheta}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos 2\pi}{2} + 0 + \frac{\sin 2\pi - \sin 0}{4}\right] = \frac{1}{\pi}\left[-\frac{\pi}{2}\right] = -\frac{1}{2}$$

$$a_1 = -\frac{1}{2} \qquad (13)$$

Therefore, by substituting values of equations (5), (11) & (13), equation (1) becomes

$$f(x) = x\sin x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = a_0 + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left\{-\frac{2\cos n\pi}{n^2-1}\right\}\cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x - \left\{\frac{2\cos 2\pi}{2^2-1}\cos 2x + \frac{2\cos 3\pi}{3^2-1}\cos 3x + \frac{2\cos 4\pi}{4^2-1}\cos 4x + \cdots\right\}$$

$$x\sin x = 1 - \frac{1}{2}\cos x - \left\{\frac{2(1)}{3}\cos 2x + \frac{2(-1)}{8}\cos 3x + \frac{2(1)}{15}\cos 4x + \cdots\right\}$$

$$x\sin x = 1 - 2\left\{\frac{\cos x}{4} + \frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} - \cdots\right\} \qquad (14)$$

Equation (14) is a Fourier series of function $x\sin x$ in the interval $-\pi \ll x \ll \pi$.

Now substitute $x = \frac{\pi}{2}$ in equation (14), we have

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 2\left\{\frac{\cos\frac{\pi}{2}}{4} + \frac{\cos 2\left(\frac{\pi}{2}\right)}{1\cdot 3} - \frac{\cos 3\left(\frac{\pi}{2}\right)}{2\cdot 4} + \frac{\cos 4\left(\frac{\pi}{2}\right)}{3\cdot 5} - \cdots\right\}$$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 2\left\{\frac{\cos\frac{\pi}{2}}{4} + \frac{\cos\pi}{1\cdot 3} - \frac{\cos\frac{3\pi}{2}}{2\cdot 4} + \frac{\cos 2\pi}{3\cdot 5} - \cdots\right\} \qquad (15)$$

As $\sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \cdots = 0$, equation (15) becomes

$$\frac{\pi}{2}(1) = 1 - 2\left\{\frac{0}{4} + \frac{(-1)}{1\cdot 3} - \frac{0}{2\cdot 4} + \frac{1}{3\cdot 5} - \cdots\right\}$$

$$\frac{\pi}{2} = 1 - 2\left\{-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7}\dots\right\} \qquad (16)$$

Now divide both sides of equation (16) with 2, we get

$$\frac{\pi}{4} = \frac{1}{2} - \left\{-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7}\dots\right\}$$

$$\therefore \quad \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots \qquad (17)$$

Which is required expression.

Example: Find a series of sines and cosines of multiplies of x, which will represent $x + x^2$ in the interval $-\pi \ll x \ll \pi$. Deduce that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Solution: Let the Fourier series of the given function $f(x) = x + x^2$ be

$$f(x) = x + x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\vartheta + \vartheta^2]\, d\vartheta \qquad (2)$$

$$= \frac{1}{2\pi}\left[\frac{\vartheta^2}{2} + \frac{\vartheta^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2 - (-\pi)^2}{2} + \frac{\pi^3 - (-\pi)^3}{3}\right] = \frac{1}{2\pi}\left[\frac{\pi^2 - \pi^2}{2} + \frac{\pi^3 + \pi^3}{3}\right] = \frac{1}{2\pi}\left[\frac{2\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$a_0 = \frac{\pi^2}{3} \qquad (3)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta = \frac{1}{\pi}\int_{-\pi}^{\pi}[\vartheta + \vartheta^2]\cos n\vartheta \; d\vartheta \qquad (4)$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \ d\vartheta + \int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \ d\vartheta\right] \quad (5)$$

We know the rule for integration by parts:

$$\int_a^b u \cdot v \, dx = \left[u \int v \, dx - \int \left\{\frac{du}{dx} \int v \, dx\right\} dx\right]_a^b \quad (6)$$

We take $u = \vartheta, v = \cos n\vartheta$ in first integral of equation (5), we have

$$\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \ d\vartheta = \left[\vartheta \int \cos n\vartheta \ d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta} \int \cos n\vartheta \ d\vartheta\right\} d\vartheta\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\vartheta \sin n\vartheta}{n} - \int \left\{\frac{\sin n\vartheta}{n}\right\} d\vartheta\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\vartheta \sin n\vartheta}{n} - \left\{-\frac{\cos n\vartheta}{n^2}\right\}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\vartheta \sin n\vartheta}{n} + \frac{\cos n\vartheta}{n^2}\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\pi \sin n\pi - (-\pi)\sin(-n\pi)}{n} + \frac{\cos n\pi - \cos(-n\pi)}{n^2}\right]$$

$$= \left[\frac{\pi \sin n\pi - \pi \sin n\pi}{n} + \frac{\cos n\pi - \cos n\pi}{n^2}\right] = 0$$

$$\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \ d\vartheta = 0 \quad (7)$$

By taking $u = \vartheta^2, v = \cos n\vartheta$ in second integral of equation (5), we have

$$\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \ d\vartheta = \left[\vartheta^2 \int \cos n\vartheta \ d\vartheta - \int \left\{\frac{d\vartheta^2}{d\vartheta} \int \cos n\vartheta \ d\vartheta\right\} d\vartheta\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\vartheta^2 \sin n\vartheta}{n} - \int \left\{2\vartheta \ \frac{\sin n\vartheta}{n}\right\} d\vartheta\right]_{-\pi}^{\pi} = \left[\frac{\vartheta^2 \sin n\vartheta}{n} - \frac{2}{n}\int \vartheta \sin n\vartheta \, d\vartheta\right]_{-\pi}^{\pi}$$

$$= \left[\frac{\pi^2 \sin n\pi - (-\pi)^2 \sin(-n\pi)}{n} - \frac{2}{n}\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \, d\vartheta\right]$$

$$\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \ d\vartheta = -\frac{2}{n}\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \, d\vartheta \quad (8)$$

By taking $u = \vartheta, v = \sin n\vartheta$

$$\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \ d\vartheta = -\frac{2}{n}\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \, d\vartheta = -\frac{2}{n}\left[\left\{\vartheta \int \sin n\vartheta \ d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta} \int \sin n\vartheta \ d\vartheta\right\} d\vartheta\right\}\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\left[-\frac{\vartheta \cos n\vartheta}{n} - \int \left\{-\frac{\cos n\vartheta}{n}\right\} d\vartheta\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\left[-\frac{\vartheta \cos n\vartheta}{n} + \frac{\sin n\vartheta}{n^2}\right]_{-\pi}^{\pi} = -\frac{2}{n}\left[-\frac{\pi \cos n\pi - (-\pi)\cos(-n\pi)}{n} + \frac{\sin n\pi - \sin(-n\pi)}{n^2}\right]$$

$$= -\frac{2}{n}\left[-\frac{\pi \cos n\pi + \pi \cos n\pi}{n}\right] = -\frac{2}{n}\left[-\frac{2\pi \cos n\pi}{n}\right] = \frac{4\pi \cos n\pi}{n^2}$$

$$\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \; d\vartheta = -\frac{2}{n}\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \, d\vartheta = \frac{4\pi \cos n\pi}{n^2} \quad (9)$$

From equation (9) we can also write as

$$\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \, d\vartheta = -\frac{2\pi \cos n\pi}{n} \quad (10)$$

This equation will be used for further calculations.

By substituting values of equation (7) and equation (9) in equation (5), we have

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \; d\vartheta + \int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \; d\vartheta\right] = \frac{1}{\pi}\left[0 + \frac{4\pi \cos n\pi}{n^2}\right]$$

$$a_n = \frac{4 \cos n\pi}{n^2} \quad (11)$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta) \sin n\vartheta \; d\vartheta = \frac{1}{\pi}\int_{-\pi}^{\pi} [\vartheta + \vartheta^2] \sin n\vartheta \; d\vartheta$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \; d\vartheta + \int_{-\pi}^{\pi} \vartheta^2 \sin n\vartheta \; d\vartheta\right] \quad (12)$$

From equation (10), we have first integral of equation (12) written as

$$\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \, d\vartheta = -\frac{2\pi \cos n\pi}{n} \quad (10)$$

By taking $u = \vartheta^2, v = \sin n\vartheta$ in second integral of equation (12), we have

$$\int_{-\pi}^{\pi} \vartheta^2 \sin n\vartheta \; d\vartheta = \left[\vartheta^2 \int \sin n\vartheta \; d\vartheta - \int \left\{\frac{d\vartheta^2}{d\vartheta} \int \sin n\vartheta \; d\vartheta\right\} d\vartheta\right]_{-\pi}^{\pi}$$

$$= \left[-\frac{\vartheta^2 \cos n\vartheta}{n} - \int \left\{-2\vartheta \; \frac{\cos n\vartheta}{n}\right\} d\vartheta\right]_{-\pi}^{\pi} = \left[-\frac{\vartheta^2 \cos n\vartheta}{n} + \frac{2}{n}\int \vartheta \cos n\vartheta \, d\vartheta\right]_{-\pi}^{\pi}$$

$$= \left[-\frac{\pi^2 \cos n\pi - (-\pi)^2 \cos(-n\pi)}{n} + \frac{2}{n}\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \, d\vartheta\right]$$

$$= \left[-\frac{\pi^2 \cos n\pi - \pi^2 \cos n\pi}{n} + \frac{2}{n}\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \, d\vartheta\right]$$

$$\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta \; d\vartheta = \frac{2}{n}\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \, d\vartheta \quad (13)$$

But from equation (7) $\int_{-\pi}^{\pi} \vartheta \cos n\vartheta \; d\vartheta = 0$, RHS of equation (13) becomes zero.

$$\therefore \int_{-\pi}^{\pi} \vartheta^2 \sin n\vartheta \; d\vartheta = 0 \quad (14)$$

By substituting values of equation (10) and (14) in equation (12), we have

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \vartheta \sin n\vartheta \; d\vartheta + \int_{-\pi}^{\pi} \vartheta^2 \sin n\vartheta \; d\vartheta\right] = \frac{1}{\pi}\left[-\frac{2\pi \cos n\pi}{n} + 0\right]$$

$$b_n = -\frac{2\cos n\pi}{n} \quad (15)$$

By rewriting equations (3), (11) and (15)

$$a_0 = \frac{\pi^2}{3}, \qquad a_n = \frac{4\cos n\pi}{n^2}, \qquad b_n = -\frac{2\cos n\pi}{n} \quad (16)$$

Substituting values of equation (16) in equation (1), we have

$$f(x) = x + x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

$$f(x) = x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4\cos n\pi}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\left[-\frac{2\cos n\pi}{n}\right]\sin nx \quad (17)$$

By expanding the summations of equation (17), we have

$$f(x) = x + x^2 = \frac{\pi^2}{3} + 4\left[\frac{\cos \pi}{1^2}\cos x + \frac{\cos 2\pi}{2^2}\cos 2x + \frac{\cos 3\pi}{3^2}\cos 3x + \frac{\cos 4\pi}{4^2}\cos 4x + \cdots\right]$$
$$- 2\left[\frac{\cos \pi}{1}\sin x + \frac{\cos 2\pi}{2}\sin 2x + \frac{\cos 3\pi}{3}\sin 3x + \frac{\cos 4\pi}{4}\sin 4x + \cdots\right]$$

$$f(x) = x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x + \cdots\right]$$
$$- 2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots\right]$$

$$f(x) = x + x^2 = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \cdots\right]$$
$$+ 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right] \quad (18)$$

At extremum π and $-\pi$, the sum of series

$$f(\pi) = \frac{1}{2}\,[f(-\pi + 0) + f(\pi - 0)] = \frac{1}{2}[-\pi + \pi^2 + \pi + \pi^2] = \pi^2$$

$$\therefore\ f(\pi) = \pi^2 \quad (19)$$

Substitute $x == \pi$ in equation (18), we have

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} - 4\left[\cos\pi - \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi - \frac{1}{4^2}\cos 4\pi + \cdots\right]$$
$$+ 2\left[\sin\pi - \frac{1}{2}\sin 2\pi + \frac{1}{3}\sin 3\pi - \frac{1}{4}\sin 4\pi + \cdots\right]$$

$$\therefore\ \pi^2 = \frac{\pi^2}{3} - 4\left[-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right]$$

$$\therefore\ \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\therefore\ \pi^2 - \frac{\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{3\pi^2 - \pi^2}{3} = \frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{2\pi^2}{3} \times \frac{1}{4} = \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad (20)$$

Which is required series.

Example: Find the series of sines and cosines of multiplies of x which represents $f(x)$ in the interval $-\pi < x < \pi$. Where

$$f(x) = 0 \ \ when\ \ -\pi < x \ll 0$$

$$= \frac{\pi x}{4}\ \ when\ 0 \le x < \pi$$

And hence deduce $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Solution: Let $f(x)$ be represented by Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

Where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(\vartheta)\, d\vartheta + \int_{0}^{\pi} f(\vartheta)\, d\vartheta\right] \quad (2)$$

$$a_0 = \frac{1}{2\pi}\left[0 + \int_0^{\pi} \frac{\pi\vartheta}{4}\ d\vartheta\right] = \frac{1}{2\pi}\left[\frac{\pi}{4}\int_0^{\pi} \vartheta\ d\vartheta\right] = \frac{1}{2\pi}\frac{\pi}{4}\left[\frac{\vartheta^2}{2}\right]_0^{\pi} = \frac{\pi^2}{16}$$

$$\therefore \quad a_0 = \frac{\pi^2}{16} \qquad (3)$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\cos n\vartheta\ d\vartheta + \int_0^{\pi} f(\vartheta)\cos n\vartheta\ d\vartheta\right]$$

$$a_n = \frac{1}{\pi}\left[0 + \int_0^{\pi} \frac{\pi\vartheta}{4}\cos n\vartheta\ d\vartheta\right] = \frac{1}{\pi}\left[\frac{\pi}{4}\int_0^{\pi} \vartheta\cos n\vartheta\ d\vartheta\right]$$

$$a_n = \frac{1}{4}\left[\int_0^{\pi} \vartheta\cos n\vartheta\ d\vartheta\right] \qquad (4)$$

By taking $u = \vartheta, v = \cos n\vartheta$ and using the rule for integration by parts:

$$\int_a^b u \cdot v\ dx = \left[u \int v\ dx - \int \left\{\frac{du}{dx}\int v\ dx\right\} dx\right]_a^b \qquad (5)$$

$$a_n = \frac{1}{4}\int_0^{\pi} \vartheta\cos n\vartheta\ d\vartheta = \frac{1}{4}\left[\vartheta\int \cos n\vartheta\ d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta}\int \cos n\vartheta\ d\vartheta\right\} d\vartheta\right]_0^{\pi}$$

$$a_n = \frac{1}{4}\left[\frac{\vartheta\sin n\vartheta}{n} - \int\left\{\frac{\sin n\vartheta}{n}\right\} d\vartheta\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\vartheta\sin n\vartheta}{n} - \left\{-\ \frac{\cos n\vartheta}{n^2}\right\}\right]_{-\pi}^{\pi} = \frac{1}{4}\left[\frac{\vartheta\sin n\vartheta}{n} + \frac{\cos n\vartheta}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\pi\sin n\pi - 0}{n} + \frac{\cos n\pi - \cos(0)}{n^2}\right] = \frac{1}{4}\left[\frac{\cos n\pi - 1}{n^2}\right]$$

$$a_n = \frac{1}{4n^2}[\cos n\pi - 1] \quad (6)$$

We know that

$$\cos n\pi = \begin{cases} -1\ for\ n = 1, 3, 5, \dots \\ \ \ 1\ for\ n = 0, 2, 4, \dots \end{cases}$$

$$\therefore \quad a_n = \frac{(-1)^n - 1}{4n^2} \quad (7)$$

Similarly, we have

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\sin n\vartheta\ d\vartheta + \int_0^{\pi} f(\vartheta)\sin n\vartheta\ d\vartheta\right]$$

$$b_n = \frac{1}{\pi}\left[0 + \int_0^{\pi} \frac{\pi\vartheta}{4}\sin n\vartheta\ d\vartheta\right] = \frac{1}{\pi}\left[\frac{\pi}{4}\int_0^{\pi} \vartheta\sin n\vartheta\ d\vartheta\right] = \frac{1}{4}\left[\int_0^{\pi} \vartheta\sin n\vartheta\ d\vartheta\right]$$

$$b_n = \frac{1}{4}\left[\int_0^{\pi} \vartheta \sin n\vartheta \ d\vartheta\right] \quad (8)$$

By taking $u = \vartheta, v = \sin n\vartheta$ and using the rule for integration by parts:

$$b_n = \frac{1}{4}\int_0^{\pi} \vartheta \sin n\vartheta \ d\vartheta = \frac{1}{4}\left[\vartheta \int \sin n\vartheta \ d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta}\int \sin n\vartheta \ d\vartheta\right\} d\vartheta\right]_0^{\pi}$$

$$= \frac{1}{4}\left[-\frac{\vartheta \cos n\vartheta}{n} - \int \left\{-\frac{\cos n\vartheta}{n}\right\} d\vartheta\right]_0^{\pi}$$

$$b_n = \frac{1}{4}\left[-\frac{\vartheta \cos n\vartheta}{n} + \frac{\sin n\vartheta}{n^2}\right]_{-\pi}^{\pi} = \frac{1}{4}\left[-\frac{\pi \cos n\pi - 0}{n} + \frac{\sin n\pi - \sin 0}{n^2}\right] = \frac{1}{4}\left[-\frac{\pi \cos n\pi}{n}\right]$$

$$b_n = -\frac{\pi \cos n\pi}{4n} = -\frac{\pi(-1)^n}{4n} \ or \ b_n = \frac{\pi(-1)^{n+1}}{4n} \quad (9)$$

$$\therefore \quad a_0 = \frac{\pi^2}{16}, \qquad a_n = \frac{(-1)^n - 1}{4n^2}, \qquad b_n = -\frac{\pi(-1)^n}{4n} = \frac{\pi(-1)^{n+1}}{4n}$$

Equation (1) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{4n^2}\right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{\pi(-1)^{n+1}}{4n}\right]\sin nx$$

By expanding the summations, we have

$$f(x) = \frac{\pi^2}{16} + \left[\frac{(-1)^1 - 1}{4\cdot 1^2}\right]\cos x + \left[\frac{(-1)^2 - 1}{4\cdot 2^2}\right]\cos 2x + \left[\frac{(-1)^3 - 1}{4\cdot 3^2}\right]\cos 3x + \left[\frac{(-1)^4 - 1}{4\cdot 4^2}\right]\cos 4x$$
$$+ \left[\frac{(-1)^5 - 1}{4\cdot 5^2}\right]\cos 5x + \cdots + \left[\frac{\pi(-1)^{1+1}}{4\cdot 1}\right]\sin x + \left[\frac{\pi(-1)^{2+1}}{4\cdot 2}\right]\sin 2x$$
$$+ \left[\frac{\pi(-1)^{3+1}}{4\cdot 3}\right]\sin 3x + \left[\frac{\pi(-1)^{4+1}}{4\cdot 4}\right]\sin 4x + \left[\frac{\pi(-1)^{5+1}}{4\cdot 5}\right]\sin 5x + \cdots$$

$$f(x) = \frac{\pi^2}{16} + \left[\frac{-2}{4}\right]\cos x + \left[\frac{-2}{4\cdot 3^2}\right]\cos 3x + \left[\frac{-2}{4\cdot 5^2}\right]\cos 5x + \cdots + \left[\frac{\pi}{4\cdot 1}\right]\sin x + \left[\frac{-\pi}{4\cdot 2}\right]\sin 2x$$
$$+ \left[\frac{\pi}{4\cdot 3}\right]\sin 3x + \left[\frac{-\pi}{4\cdot 4}\right]\sin 4x + \left[\frac{\pi}{4\cdot 5}\right]\sin 5x + \cdots$$

$$f(x) = \frac{\pi^2}{16} - \frac{1}{2}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \frac{\pi}{4}\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right] \quad (10)$$

This is a Fourier series of a given function $f(x)$.

At extremum π and $-\pi$, the sum of series

$$f(\pi) = \frac{1}{2}\ [f(-\pi + 0) + f(\pi - 0)] = \frac{1}{2}\left[0 + \left(\frac{\pi x}{4}\right)_{x=\pi}\right] = \frac{1}{2}\left[\frac{\pi^2}{4}\right] = \frac{\pi^2}{8}$$

$$\therefore\ f(\pi) = \frac{\pi^2}{8} \qquad (11)$$

Substitute $x = \pi$ in equation (10), we have

$$f(\pi) = \frac{\pi^2}{8} = \frac{\pi^2}{16} - \frac{1}{2}\left[\cos\pi + \frac{\cos 3\pi}{3^2} + \frac{\cos 5\pi}{5^2} + \cdots\right] + \frac{\pi}{4}\left[\sin\pi - \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} - \frac{\sin 4\pi}{4} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{\pi^2}{16} - \frac{1}{2}\left[-1 - \frac{1}{3^2} - \frac{1}{5^2} - \cdots\right]$$

$$\frac{\pi^2}{8} - \frac{\pi^2}{16} = \frac{1}{2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{16} = \frac{1}{2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \qquad (12)$$

Which is required series.

Example: Find Fourier series for $f(x)$ in the interval $(-\pi, \pi)$, where

$$f(x) = \begin{matrix} \pi + x, when - \pi < x < 0 \\ \pi - x, when \quad 0 < x < \pi \end{matrix}$$

Solution: Let $f(x)$ be represented by Fourier series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

Where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\ d\vartheta = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(\vartheta)\ d\vartheta + \int_{0}^{\pi} f(\vartheta)\ d\vartheta\right] \quad (2)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} (\pi + \vartheta) d\vartheta + \int_{0}^{\pi} (\pi - \vartheta) d\vartheta\right]$$

$$= \frac{1}{2\pi}\left[\left\{\pi\vartheta + \frac{\vartheta^2}{2}\right\}_{-\pi}^{0} + \left\{\pi\vartheta - \frac{\vartheta^2}{2}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[0 - \left\{\pi(-\pi) + \frac{(-\pi)^2}{2}\right\} + \left\{\pi(\pi) - \frac{(\pi)^2}{2}\right\} - 0\right] = \frac{1}{2\pi}\left[\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right] = \frac{1}{2\pi}[\pi^2]$$

$$\therefore\ a_0 = \frac{\pi}{2} \quad (3)$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\cos n\vartheta\ d\vartheta + \int_{0}^{\pi} f(\vartheta)\cos n\vartheta\ d\vartheta\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi+\vartheta)\cos n\vartheta\ d\vartheta + \int_{0}^{\pi} (\pi-\vartheta)\cos n\vartheta\ d\vartheta\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi\cos n\vartheta\ d\vartheta + \int_{-\pi}^{0} \vartheta\cos n\vartheta\ d\vartheta + \int_{0}^{\pi} \pi\cos n\vartheta\ d\vartheta - \int_{0}^{\pi} \vartheta\cos n\vartheta\ d\vartheta\right] \quad (4)$$

We calculate each term separately,

$$\int_{-\pi}^{0} \pi\cos n\vartheta\ d\vartheta = \pi\left\{\frac{\sin n\vartheta}{n}\right\}_{-\pi}^{0} = \pi\left\{\frac{\sin 0 - \sin(-n\pi)}{n}\right\} = 0 \quad (5)$$

$$\int_{0}^{\pi} \pi\cos n\vartheta\ d\vartheta = \pi\left\{\frac{\sin n\vartheta}{n}\right\}_{0}^{\pi} = \pi\left\{\frac{\sin n\pi - \sin 0}{n}\right\} = 0 \quad (6)$$

By taking $u = \vartheta, v = \cos n\vartheta$ and using the rule for integration by parts in second and fourth integral, we have:

$$\int_{a}^{b} u\cdot v\ dx = \left[u\int v\ dx - \int\left\{\frac{du}{dx}\int v\ dx\right\}dx\right]_{a}^{b} \quad (7)$$

$$\int_{-\pi}^{0} \vartheta\cos n\vartheta\ d\vartheta = \left\{\vartheta\int\cos n\vartheta\ d\vartheta - \int\left\{\frac{d\vartheta}{d\vartheta}\int\cos n\vartheta\ d\vartheta\right\}d\vartheta\right\}_{-\pi}^{0}$$

$$= \left\{\vartheta\left(\frac{\sin n\vartheta}{n}\right) - \int\left\{\frac{\sin n\vartheta}{n}\right\}d\vartheta\right\}_{-\pi}^{0} = \left\{\frac{\vartheta\sin n\vartheta}{n} - \left(-\frac{\cos n\vartheta}{n^2}\right)\right\}_{-\pi}^{0} = \left\{\frac{\vartheta\sin n\vartheta}{n} + \frac{\cos n\vartheta}{n^2}\right\}_{-\pi}^{0}$$

$$= \frac{0 - (-\pi)\sin(-n\pi)}{n} + \frac{\cos 0 - \cos(-n\pi)}{n^2} = \frac{1 - \cos n\pi}{n^2} \quad (8)$$

$$\int_{0}^{\pi} \vartheta\cos n\vartheta\ d\vartheta = \left\{\frac{\vartheta\sin n\vartheta}{n} + \frac{\cos n\vartheta}{n^2}\right\}_{0}^{\pi} = \frac{\pi\sin n\pi - 0}{n} + \frac{\cos n\pi - \cos 0}{n^2} = \frac{\cos n\pi - 1}{n^2} \quad (9)$$

By substituting values of equations (5), (6), (8) & (9) in equation (4), we have

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi\cos n\vartheta\ d\vartheta + \int_{-\pi}^{0} \vartheta\cos n\vartheta\ d\vartheta + \int_{0}^{\pi} \pi\cos n\vartheta\ d\vartheta - \int_{0}^{\pi} \vartheta\cos n\vartheta\ d\vartheta\right]$$

$$a_n = \frac{1}{\pi}\left[0 + \frac{1-\cos n\pi}{n^2} + 0 - \frac{\cos n\pi - 1}{n^2}\right] = \frac{2}{\pi n^2}[1 - \cos n\pi] = \frac{2}{\pi n^2}[1 - (-1)^n]$$

$$a_n = \frac{2}{\pi n^2}[1-(-1)^n] \qquad (10)$$

Similarly, we have

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\sin n\vartheta \; d\vartheta + \int_{0}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi+\vartheta)\sin n\vartheta \; d\vartheta + \int_{0}^{\pi} (\pi-\vartheta)\sin n\vartheta \; d\vartheta\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi\sin n\vartheta \; d\vartheta + \int_{-\pi}^{0} \vartheta\sin n\vartheta \; d\vartheta + \int_{0}^{\pi} \pi\sin n\vartheta \; d\vartheta - \int_{0}^{\pi} \vartheta\sin n\vartheta \; d\vartheta\right] \qquad (11)$$

We calculate each term separately,

$$\int_{-\pi}^{0} \pi\sin n\vartheta \; d\vartheta = \pi\left\{-\frac{\cos n\vartheta}{n}\right\}_{-\pi}^{0} = \pi\left\{-\frac{\cos 0-\cos(-n\pi)}{n}\right\} = \frac{\pi}{n}(\cos n\pi - 1) \qquad (12)$$

$$\int_{0}^{\pi} \pi\sin n\vartheta \; d\vartheta = \pi\left\{-\frac{\cos n\vartheta}{n}\right\}_{0}^{\pi} = \pi\left\{-\frac{\cos n\pi-\cos 0}{n}\right\} = \frac{\pi}{n}(1-\cos n\pi) \qquad (13)$$

By taking $u=\vartheta, v=\cos n\vartheta$ and using the rule for integration by parts in second and fourth integral, we have:

$$\int_{-\pi}^{0} \vartheta\sin n\vartheta \; d\vartheta = \left\{\vartheta\int \sin n\vartheta \; d\vartheta - \int\left\{\frac{d\vartheta}{d\vartheta}\int \sin n\vartheta \; d\vartheta\right\} d\vartheta\right\}_{-\pi}^{0}$$

$$= \left\{\vartheta\left(-\frac{\cos n\vartheta}{n}\right) - \int\left\{-\frac{\cos n\vartheta}{n}\right\} d\vartheta\right\}_{-\pi}^{0} = \left\{-\frac{\vartheta\cos n\vartheta}{n} + \frac{\sin n\vartheta}{n^2}\right\}_{-\pi}^{0}$$

$$= -\frac{0-(-\pi)\cos(-n\pi)}{n} + \frac{\sin 0 - \sin(-n\pi)}{n^2} = -\frac{\pi\cos n\pi}{n} \qquad (14)$$

$$\int_{0}^{\pi} \vartheta\sin n\vartheta \; d\vartheta = \left\{-\frac{\vartheta\cos n\vartheta}{n} + \frac{\sin n\vartheta}{n^2}\right\}_{0}^{\pi} = -\frac{\pi\cos n\pi - 0}{n} + \frac{\sin n\pi - \sin 0}{n^2} = -\frac{\pi\cos n\pi}{n} \qquad (15)$$

By substituting values of equations (12), (13), (14) & (15) in equation (11), we have

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi\sin n\vartheta \; d\vartheta + \int_{-\pi}^{0} \vartheta\sin n\vartheta \; d\vartheta + \int_{0}^{\pi} \pi\sin n\vartheta \; d\vartheta - \int_{0}^{\pi} \vartheta\sin n\vartheta \; d\vartheta\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos n\pi - 1) + \frac{\pi}{n}(1-\cos n\pi) - \frac{\pi\cos n\pi}{n} - \left(-\frac{\pi\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{n}[\cos n\pi - 1 + 1 - \cos n\pi - \cos n\pi + \cos n\pi] = 0$$

$$\therefore \;\; b_n = 0 \qquad (16)$$

$$\therefore \quad a_0 = \frac{\pi}{2}, \qquad a_n = \frac{2}{\pi n^2}[1-(-1)^n], \qquad b_n = 0 \quad (17)$$

Equation (1) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[1-(-1)^n]\cos nx = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos nx$$

By expanding the summations, we have

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[\begin{array}{c}\frac{[1-(-1)]}{1^2}\cos x + \frac{[1-(-1)^2]}{2^2}\cos 2x + \frac{[1-(-1)^3]}{3^2}\cos 3x \\ + \frac{[1-(-1)^4]}{4^2}\cos 4x + \frac{[1-(-1)^5]}{5^2}\cos 5x + \cdots\end{array}\right]$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\left[2\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \cdots\right]$$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] \quad (18)$$

This is a Fourier series of a given function $f(x)$.

Example: Obtain the Fourier Series for a function $f(x)$, where

$$f(x) = \begin{array}{cc} \cos x & for \quad 0 \le x \le \pi \\ -\cos x & for -\pi \le x \le 0 \end{array}$$

Solution: Let the Fourier series represented by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(\vartheta)\, d\vartheta + \int_{0}^{\pi} f(\vartheta)\, d\vartheta\right]$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-cos\vartheta)\, d\vartheta + \int_{0}^{\pi} \cos\vartheta \;\, d\vartheta\right]$$

$$a_0 = \frac{1}{2\pi}\left[-\{\sin\vartheta\}_{-\pi}^{0} + \{\sin\vartheta\}_{0}^{\pi}\right] = \frac{1}{2\pi}\left[-\{\sin 0 - \sin(-\pi)\} + \{\sin\pi - \sin 0\}\right] = 0$$

$$\therefore \quad a_0 = 0 \quad (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta\, d\vartheta$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\cos n\vartheta \; d\vartheta + \int_{0}^{\pi} f(\vartheta)\cos n\vartheta \; d\vartheta\right]$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos\vartheta \cos n\vartheta \; d\vartheta + \int_{0}^{\pi} \cos\vartheta \cos n\vartheta \; d\vartheta\right] \quad (3)$$

We know that $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$\therefore\; a_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \frac{1}{2}\{\cos(n+1)\vartheta + \cos(n-1)\vartheta\}\, d\vartheta + \int_{0}^{\pi} \frac{1}{2}\{\cos(n+1)\vartheta + \cos(n-1)\vartheta\}\, d\vartheta\right]$$

$$a_n = \frac{1}{2\pi}\left[-\left\{\frac{\sin(n+1)\vartheta}{n+1} + \frac{\sin(n-1)\vartheta}{n-1}\right\}_{-\pi}^{0} + \left\{\frac{\sin(n+1)\vartheta}{n+1} + \frac{\sin(n-1)\vartheta}{n-1}\right\}_{0}^{\pi}\right] = 0$$

$$\therefore\; a_n = 0 \quad (4)$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\sin n\vartheta \; d\vartheta + \int_{0}^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta\right]$$

$$b_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos\vartheta \sin n\vartheta \; d\vartheta + \int_{0}^{\pi} \cos\vartheta \sin n\vartheta \; d\vartheta\right] \quad (5)$$

We know that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$\therefore b_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \frac{1}{2}\{\sin(n+1)\vartheta + \sin(n-1)\vartheta\}\, d\vartheta + \int_{0}^{\pi} \frac{1}{2}\{\sin(n+1)\vartheta + \sin(n-1)\vartheta\}\, d\vartheta\right]$$

$$\therefore b_n = \frac{1}{2\pi}\left[-\left\{-\frac{\cos(n+1)\vartheta}{n+1} - \frac{\cos(n-1)\vartheta}{n-1}\right\}_{-\pi}^{0} + \left\{-\frac{\cos(n+1)\vartheta}{n+1} - \frac{\cos(n-1)\vartheta}{n-1}\right\}_{0}^{\pi}\right]$$

$$\therefore b_n = \frac{1}{2\pi}\left[\frac{\cos 0 - \cos(n+1)(-\pi)}{n+1} + \frac{\cos 0 - \cos(n-1)(-\pi)}{n-1} - \frac{\cos(n+1)\pi - \cos 0}{n+1} - \frac{\cos(n-1)\pi - \cos 0}{n-1}\right]$$

$$\therefore b_n = \frac{1}{2\pi}\left[\frac{1-\cos(n+1)\pi}{n+1} + \frac{1-\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi - 1}{n+1} - \frac{\cos(n-1)\pi - 1}{n-1}\right]$$

$$\therefore b_n = \frac{1}{2\pi}\left[\frac{2-2\cos(n+1)\pi}{n+1} + \frac{2-2\cos(n-1)\pi}{n-1}\right]$$

As $\cos(\pi+\theta) = -\cos\theta\,, \cos(\pi-\theta) = -\cos\theta \;\therefore\; \cos(n+1)\pi = -\cos n\pi\,, \cos(n-1)\pi = -\cos n\pi$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right] = \frac{1}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)[1+\cos n\pi]$$

$$= \frac{1}{\pi}\left(\frac{n-1+n+1}{n^2-1}\right)[1+\cos n\pi] = \frac{1}{\pi}\left(\frac{2n}{n^2-1}\right)[1+\cos n\pi]$$

$$\therefore b_n = \frac{2n}{\pi(n^2-1)}[1+\cos n\pi]$$

$$\therefore b_n = \frac{2n}{\pi(n^2-1)}[1+(-1)^n] \quad (5)$$

Above equation is true for $n \neq 1$. Because for $n = 1, b_n = b_1 = \frac{0}{0}$, which is indefinite.

For $n = 1$, equation (5) becomes

$$b_1 = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos\vartheta \sin\vartheta \;d\vartheta + \int_{0}^{\pi} \cos\vartheta \sin\vartheta \;d\vartheta\right] = \frac{1}{2\pi}\left[-\int_{-\pi}^{0} \sin 2\vartheta \;d\vartheta + \int_{0}^{\pi} \sin 2\vartheta \;d\vartheta\right]$$

(As $2\sin\theta\cos\theta = \sin 2\theta$)

$$b_1 = \frac{1}{2\pi}\left[-\left\{-\frac{\cos 2\vartheta}{2}\right\}_{-\pi}^{0} + \left\{-\frac{\cos 2\vartheta}{2}\right\}_{0}^{\pi}\right] = \frac{1}{4\pi}\left[\{\cos 2\vartheta\}_{-\pi}^{0} - \{\cos 2\vartheta\}_{0}^{\pi}\right]$$

$$= \frac{1}{4\pi}[\{\cos 0 - \cos(-2\pi)\} - \{\cos 2\pi - \cos 0\}] = \frac{1}{4\pi}[\{1-1\} - \{1-1\}] = 0$$

$$\therefore \quad a_0 = 0, \qquad a_n = 0, \qquad b_1 = 0, \qquad b_n = \frac{2n}{\pi(n^2-1)}[1+(-1)^n] \qquad (6)$$

Equation (1) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$f(x) = 0 + \sum_{n=2}^{\infty} \frac{2n}{\pi(n^2-1)}[1+(-1)^n] \sin nx = \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{n[1+(-1)^n]}{(n^2-1)} \sin nx$$

On expanding the summation, we have

$$f(x) = \frac{2}{\pi}\left[\frac{2[1+(-1)^2]}{(2^2-1)}\sin 2x + \frac{3[1+(-1)^3]}{(3^2-1)}\sin 3x + \frac{4[1+(-1)^4]}{(4^2-1)}\sin 4x \right.$$
$$\left. + \frac{5[1+(-1)^5]}{(5^2-1)}\sin 5x + \frac{6[1+(-1)^6]}{(6^2-1)}\sin 6x + \cdots\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{4}{3}\sin 2x + \frac{8}{15}\sin 4x + \frac{12}{35}\sin 6x + \cdots\right]$$

$$f(x) = \frac{4}{\pi}\left[\frac{2}{1\cdot 3}\sin 2x + \frac{4}{3\cdot 5}\sin 4x + \frac{6}{5\cdot 7}\sin 6x + \cdots\right] \quad (7)$$

Which is the required series.

Example: Find the values of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ using Fourier series.

Solution: Let $f(x) = x^2$

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

As the function is even function, $f(-x) = (-x)^2 = x^2 = f(x)$, $b_n = 0$ (2)

Therefore equation (1) becomes,

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (3)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \vartheta^2\, d\vartheta = \frac{1}{2\pi}\left[\frac{\vartheta^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^3-(-\pi)^3}{3}\right] = \frac{1}{2\pi}\left[\frac{2\pi^3}{3}\right] = \frac{\pi^2}{3} \qquad (4)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta\ d\vartheta = \frac{1}{\pi}\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta\ d\vartheta$$

We have derived equation (9) of example (2) as,

$$\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta\ d\vartheta = -\frac{2}{n}\int_{-\pi}^{\pi} \vartheta \sin n\vartheta\, d\vartheta = \frac{4\pi\cos n\pi}{n^2}$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \vartheta^2 \cos n\vartheta\ d\vartheta = \frac{1}{\pi}\left[\frac{4\pi\cos n\pi}{n^2}\right] = \frac{4\cos n\pi}{n^2} = (-1)^n\frac{4}{n^2} \qquad (5)$$

Therefore equation (3) becomes,

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[(-1)^n\frac{4}{n^2}\right]\cos nx \qquad (6)$$

By expanding the summation of equation (6), we have

$$x^2 = \frac{\pi^2}{3} + (-1)\frac{4}{1^2}\cos x + (-1)^2\frac{4}{2^2}\cos 2x + (-1)^3\frac{4}{3^2}\cos 3x + (-1)^4\frac{4}{4^2}\cos 4x + \cdots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x \dots\right] \qquad (7)$$

For $x = 0$, equation (7) becomes

$$0 = \frac{\pi^2}{3} - 4\left[\cos 0 - \frac{1}{2^2}\cos 0 + \frac{1}{3^2}\cos 0 - \frac{1}{4^2}\cos 0 \dots\right]$$

$$0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots\right]$$

$$\therefore 0 = \frac{\pi^2}{3} - 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots - \frac{2}{2^2} - \frac{2}{4^2} - \cdots\right]$$

$$\therefore 0 = \frac{\pi^2}{3} - 4\left[\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots\right\} - \frac{2}{2^2}\left\{1 + \frac{1}{2^2} + \cdots\right\}\right]$$

$$0 = \frac{\pi^2}{3} - 4\left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}\right] = \frac{\pi^2}{3} - 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{3} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \therefore \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Which is required expression.

Example: Using Fourier series prove that:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

The LHS of above equation is expanding as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} \dots$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \dots - \frac{1}{2^2} - \frac{1}{4^2} - \cdots$$

$$= \left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \dots\right\} - \frac{1}{2^2}\left\{1 + \frac{1}{2^2} + \cdots\right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Example: Find a series of cosines of multiplies of x which represents x in the interval $(0, \pi)$. Hence deduce that:

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} \dots$$

Draw graph of the function.

Solution: Let $f(x) = x$ be the function. The Fourier series is given as

$$f(x) = x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{\pi}\int_0^{\pi} \vartheta\, d\vartheta = \frac{1}{\pi}\left[\frac{\vartheta^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = \frac{\pi}{2} \qquad (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\vartheta)\cos n\vartheta\ d\vartheta = \frac{2}{\pi}\int_0^{\pi} \vartheta \cos n\vartheta\ d\vartheta$$

By taking $u = \vartheta, v = \cos n\vartheta$ and using rule for integration by parts, we have

$$a_n = \frac{2}{\pi}\left[\vartheta \int \cos n\vartheta\ d\vartheta - \int\left\{\frac{d\vartheta}{d\vartheta}\int \cos n\vartheta\ d\vartheta\right\} d\vartheta\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\vartheta \sin n\vartheta}{n} - \int\left\{\frac{\sin n\vartheta}{n}\right\} d\vartheta\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\vartheta \sin n\vartheta}{n} + \frac{\cos n\vartheta}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi \sin n\pi - 0}{n} + \frac{\cos n\pi - \cos 0}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$\therefore\ a_n = -\frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] \qquad (3)$$

Therefore equation (1) becomes,

$$f(x) = x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx \qquad (4)$$

On expanding the summation in equation (4), we have

$$x = \frac{\pi}{2} - \frac{2}{\pi}\left[\begin{array}{c}\left\{\frac{1-(-1)}{1^2}\right\}\cos x + \left\{\frac{1-(-1)^2}{2^2}\right\}\cos 2x + \left\{\frac{1-(-1)^3}{3^2}\right\}\cos 3x \\ + \left\{\frac{1-(-1)^4}{4^2}\right\}\cos 4x + \left\{\frac{1-(-1)^3}{5^2}\right\}\cos 5x + \cdots\end{array}\right]$$

$$x = \frac{\pi}{2} - \frac{2}{\pi}\left[2\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \cdots\right]$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] \quad (5)$$

By taking $x = 0$ in equation (5), we have

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 \ldots\right] = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

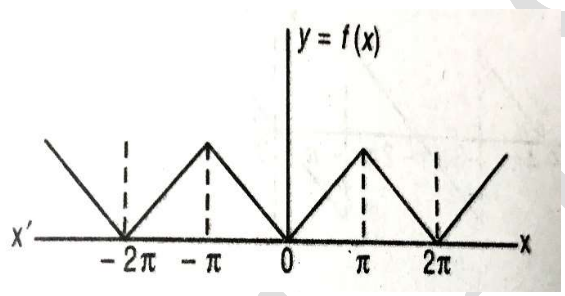
$$\frac{\pi}{2} = \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

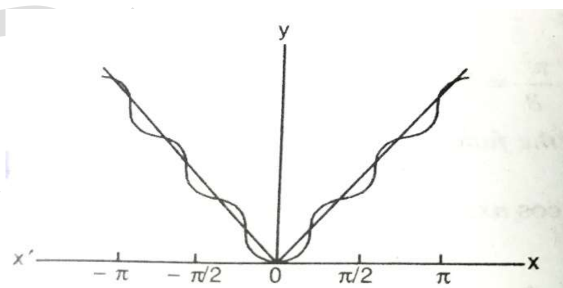
$$\frac{\pi^2}{8} = \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \quad (6)$$

This is the required series. Equation (5) is rewritten for $y = x$ as

$$y = x = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

In the interval $(0, \pi)$, the line $y = x$ gives the curves represented by the series. Hence $f(x)$ represented by the above Fourier series contains cosine terms only. So, this is an even function and the curve is symmetrical about the axis of y along which $f(x)$ is plotted. Period of the series is 2π, hence the portion between $-\pi$ to π, repeats indefinitely on both sides and the sum is continuous for all values of x.





In fact, the graph of the sum of n terms of Fourier series for $f(x)$ approximates to the graph of $f(x)$ the greater value of n is, the closer is the approximation. With three terms in equation (5), the graph is as shown in figure (2).

Example: Find the series of sines of multiples of x which represents x in the interval $\pi \geq x \geq 0$. Show by a graph the nature of the series.

Let $f(x) = x$ be a function.

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(\vartheta)\sin n\vartheta \; d\vartheta = \frac{2}{\pi}\int_0^{\pi} \vartheta \sin n\vartheta \; d\vartheta \quad (2)$$

By taking $u = \vartheta, v = \sin n\vartheta$ and using rule of integration by parts.

$$b_n = \frac{2}{\pi}\left[\vartheta \int \sin n\vartheta \; d\vartheta - \int \left\{\frac{d\vartheta}{d\vartheta}\int \sin n\vartheta \; d\vartheta\right\} d\vartheta\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{\vartheta\cos n\vartheta}{n} - \int\left\{-\frac{\cos n\vartheta}{n}\right\} d\vartheta\right]_0^\pi = \frac{2}{\pi}\left[-\frac{\vartheta\cos n\vartheta}{n} + \frac{\sin n\vartheta}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi - 0}{n} + \frac{\sin n\pi - \sin 0}{n^2}\right] = \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{2}{n}\;[\cos n\pi]$$

$$\therefore \; b_n = -\frac{2}{n}\,(-1)^n = \frac{2}{n}\,(-1)^{n+1} \quad (3)$$

Therefore equation (1) becomes

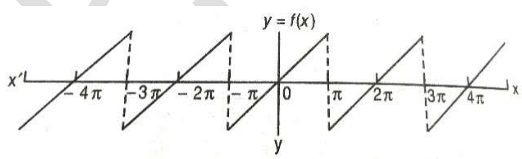
$$x = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\frac{2}{n}\,(-1)^{n+1}\sin nx = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$$

By expanding the summation, we have

$$x = 2\left[\frac{(-1)^{1+1}\sin x}{1} + \frac{(-1)^{2+1}\sin 2x}{2} + \frac{(-1)^{3+1}\sin 3x}{3} + \cdots\right]$$

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right] \quad (4)$$

This sum is discontinuous at $x = \pi$. When we draw a graph, the curve is symmetrical about the origin. The series represented between $(-\pi, \pi)$ repeat identically in both the directions. The points $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are points of discontinuity.



Example: Find the Fourier series for the periodic function $f(x)$ defined by

$$f(x) \quad \begin{aligned} &= -\pi, if \; -\pi < x < 0 \\ &= \;\; x, if \;\; 0 < x < \pi \end{aligned}$$

Hence prove that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} \ldots.$

Solution: Let $f(x)$ be a function having Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(\vartheta)\, d\vartheta + \int_0^{\pi} f(\vartheta)\, d\vartheta\right] \qquad (2)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)d\vartheta + \int_{0}^{\pi}\vartheta\, d\vartheta\right] = \frac{1}{2\pi}\left[\{-\pi\vartheta\}_{-\pi}^{0} + \left\{\frac{\vartheta^2}{2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[0 - \pi^2 + \frac{\pi^2}{2} - 0\right] = \frac{1}{2\pi}\left[-\frac{\pi^2}{2}\right] = -\frac{\pi}{4} \qquad (3)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta\ d\vartheta = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\cos n\vartheta\ d\vartheta + \int_{0}^{\pi} f(\vartheta)\cos n\vartheta\ d\vartheta\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos n\vartheta\ d\vartheta + \int_{0}^{\pi}\vartheta\cos n\vartheta\ d\vartheta\right]$$

$$= \frac{1}{\pi}\left[\left\{(-\pi)\frac{\sin n\vartheta}{n}\right\}_{-\pi}^{0} + \left\{\vartheta\int\cos n\vartheta\ d\vartheta - \int\left\{\frac{d\vartheta}{d\vartheta}\int\cos n\vartheta\ d\vartheta\right\}d\vartheta\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{(-\pi)\frac{\sin 0 - \sin(-n\pi)}{n}\right\} + \left\{\frac{\vartheta\sin n\vartheta}{n} - \int\left\{\frac{\sin n\vartheta}{n}\right\}d\vartheta\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\vartheta\sin n\vartheta}{n} + \frac{\cos n\vartheta}{n^2}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi\sin n\pi - 0}{n} + \frac{\cos n\pi - \cos 0}{n^2}\right]$$

$$\therefore\ a_n = \frac{1}{\pi n^2}[\cos n\pi - 1] \qquad (4)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\sin n\vartheta\ d\vartheta = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\vartheta)\sin n\vartheta\ d\vartheta + \int_{0}^{\pi} f(\vartheta)\sin n\vartheta\ d\vartheta\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin n\vartheta\ d\vartheta + \int_{0}^{\pi}\vartheta\sin n\vartheta\ d\vartheta\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left\{-\frac{\cos n\vartheta}{n}\right\}_{-\pi}^{0} + \left\{\vartheta\int\sin n\vartheta\ d\vartheta - \int\left\{\frac{d\vartheta}{d\vartheta}\int\sin n\vartheta\ d\vartheta\right\}d\vartheta\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{\cos 0 - \cos(-n\pi)}{n}\right\} + \left\{-\frac{\vartheta\cos n\vartheta}{n} - \int\left\{-\frac{\cos n\vartheta}{n}\right\}d\vartheta\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{1 - \cos n\pi}{n}\right\} + \left\{-\frac{\vartheta\cos n\vartheta}{n} + \frac{\sin n\vartheta}{n^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi - \pi\cos n\pi}{n}\right\} + \left\{-\frac{\pi\cos n\pi - 0}{n} + \frac{\sin n\pi - \sin 0}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi - \pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right] = \frac{1}{\pi}\left[\frac{\pi - 2\pi\cos n\pi}{n}\right]$$

$$b_n = \frac{1}{n}[1 - 2\cos n\pi] \qquad (5)$$

Therefore equation (1) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2} [\cos n\pi - 1] \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} [1 - 2\cos n\pi] \sin nx$$

By expansion of summations, we have

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\left[\frac{(\cos\pi - 1)}{1^2}\cos x + \frac{(\cos 2\pi - 1)}{2^2}\cos 2x + \frac{(\cos 3\pi - 1)}{3^2}\cos 3x + \cdots\right] + \left[\frac{(1 - 2\cos\pi)}{1}\sin x + \frac{(1 - 2\cos 2\pi)}{2}\sin 2x + \frac{(1 - 2\cos 3\pi)}{3}\sin 3x + \cdots\right]$$

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\left[-2\cos x + \frac{(-2)\cos 3x}{3^2} + \cdots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \cdots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right] \quad (6)$$

Which is the required series.

One discontinuity occurs at $x = 0$

$$f(0) = \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)] = \frac{1}{2}[-\pi + 0] = -\frac{\pi}{2} \quad (7)$$

By substituting $x = 0$ in equation (6)

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{\cos 0}{3^2} + \cdots\right] + \left[3\sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} + \cdots\right]$$

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$-\frac{\pi}{2} + \frac{\pi}{4} = -\frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$-\frac{\pi}{4} = -\frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$-\frac{\pi}{4}\left(-\frac{\pi}{2}\right) = \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Which is required expression.

Physical Applications of Fourier series:

(1) Fourier series involving phase angles:

We have

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\omega t + \sum_{n=1}^{\infty} \beta_n \sin n\omega t \qquad (1)$$

Where

$$\alpha_n = \frac{2}{\tau}\int_0^{\tau} f(t)\ \cos n\omega t\, dt\,, \qquad \beta_n = \frac{2}{\tau}\int_0^{\tau} f(t)\ \sin n\omega t\ dt\,, \qquad \tau = \frac{2\pi}{\omega} \quad (2)$$

Let

$$\alpha_n \cos n\omega t + \beta_n \sin n\omega t = \gamma_n \cos(n\omega t - \emptyset_n) \qquad (3)$$

Here $\emptyset_n$ being the phase angle. We know that $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\alpha_n \cos n\omega t + \beta_n \sin n\omega t = \gamma_n \cos n\omega t \cos \emptyset_n + \gamma_n \sin n\omega t \sin \emptyset_n \quad (4)$$

Equating coefficients of $\cos n\omega t$ and $\sin n\omega t$ on both sides of equation (4), we get

$$\alpha_n = \gamma_n \cos \emptyset_n \qquad (5)$$

$$\beta_n = \gamma_n \sin \emptyset_n \qquad (6)$$

By squaring equations (5), (6) and adding the results, we get

$$\alpha_n{}^2 + \beta_n{}^2 = \gamma_n{}^2[\cos^2 \emptyset_n + \sin^2 \emptyset_n] = \gamma_n{}^2[1]$$

$$\therefore\ \gamma_n{}^2 = \alpha_n{}^2 + \beta_n{}^2$$

$$\gamma_n = \sqrt{\alpha_n{}^2 + \beta_n{}^2} \qquad (7)$$

By taking ratio of equation (6) to equation (5), we get

$$\frac{\beta_n}{\alpha_n} = \frac{\gamma_n \sin \emptyset_n}{\gamma_n \cos \emptyset_n} = \frac{\sin \emptyset_n}{\cos \emptyset_n} = \tan \emptyset_n$$

$$\therefore\ \emptyset_n = \tan^{-1}\left(\frac{\beta_n}{\alpha_n}\right) \qquad (8)$$

Therefore equation (1) now becomes,

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\omega t + \sum_{n=1}^{\infty} \beta_n \sin n\omega t = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} [\alpha_n \cos n\omega t + \beta_n \sin n\omega t]$$

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \gamma_n \cos(n\omega t - \emptyset_n) \quad (9)$$

Or

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \gamma_n \sin\left(n\omega t + \frac{\pi}{2} - \emptyset_n\right) \quad (10)$$

(2) Effective values and the average of a product:

When dealing with the problems in electrical circuit theory and in the theory of mechanical vibrations, we require to find the root mean square of effective value of a periodic function. In terms of complex Fourier series expansion, a periodic function $f(t)$ is given by:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t} \quad (1)$$

Where $\tau = \frac{2\pi}{\omega}$

The rms or effective value of the function f over a period τ is given by

$$f_E^2 = \frac{1}{\tau}\int_0^{\tau} f^2(t)dt$$

(We know that for complex function like $z = x + iy, z^* = x - iy$,

$z^2 = z \cdot z^* = (x + iy) \cdot (x - iy) = x^2 + (-i)^2 y^2 = x^2 + y^2, \therefore\ z^2 = x^2 + y^2$)

$$f_E^2 = \frac{1}{\tau}\int_0^{\tau}\left[\sum_{n=-\infty}^{\infty} a_n e^{in\omega t} \cdot \sum_{m=-\infty}^{\infty} a_m e^{im\omega t}\right]dt$$

$$= \frac{1}{\tau}\int_0^{\tau}\left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n\, a_m\, e^{i(n+m)\omega t}\right]dt$$

$$\therefore\ f_E^2 = \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n\, a_m\left[\int_0^{\tau} e^{i(n+m)\omega t}dt\right] \quad (2)$$

Here we calculate the integral of equation (2) separately. If $n + m \neq 0$ then

$$\int_0^{\tau} e^{i(n+m)\omega t}dt = \left[\frac{e^{i(n+m)\omega t}}{i(n+m)\omega}\right]_0^{\tau} = \left[\frac{e^{i(n+m)\omega\tau} - e^0}{i(n+m)\omega}\right] = \left[\frac{e^{i(n+m)2\pi} - 1}{i(n+m)\omega}\right]$$

$$= \left[\frac{\cos(n+m)2\pi + \sin(n+m)2\pi - 1}{i(n+m)\omega}\right] = \left[\frac{1 + 0 - 1}{i(n+m)\omega}\right] = 0$$

$$\int_0^\tau e^{i(n+m)\omega t}dt = 0\ for\ n+m \neq 0 \quad (3)$$

For $n+m=0$, we have

$$\int_0^\tau e^{i(n+m)\omega t}dt = \int_0^\tau e^0 dt = \int_0^\tau dt = [t]_0^\tau = \tau \quad (4)$$

$$\therefore \int_0^\tau e^{i(n+m)\omega t}dt \begin{array}{l} = 0\ for\ m+n \neq 0 \\ = \tau\ for\ m+n = 0 \end{array} \quad (5)$$

All the integrals in equation (2) vanish except for $n+m=0$ or $m=-n$. Therefore equation (2) becomes

$$\therefore f_E^2 = \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n\, a_m \left[\int_0^\tau e^{i(n+m)\omega t}dt\right]$$

$$\therefore f_E^2 = \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\left[+\cdots + a_n\, a_{-n}\left\{\int_0^\tau e^{i(n-n)\omega t}dt\right\} + \cdots + a_n\, a_{-1}\left\{\int_0^\tau e^{i(n-1)\omega t}dt\right\}\right.$$
$$\left. + a_n\, a_0\left\{\int_0^\tau e^{i(n+0)\omega t}dt\right\} + a_n\, a_1\left\{\int_0^\tau e^{i(n+1)\omega t}dt\right\} + \cdots\right]$$

$$f_E^2 = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} a_n\, a_{-n}\tau$$

$$\therefore f_E^2 = \sum_{n=-\infty}^{\infty} a_n\, a_{-n} \quad (6)$$

By expanding the summation

$$f_E^2 = \sum_{n=-\infty}^{-1} a_n\, a_{-n} + a_0^2 + \sum_{n=1}^{\infty} a_n\, a_{-n}$$

$$f_E^2 = a_0^2 + 2\sum_{n=1}^{\infty} |a_n|^2 \quad (7)$$

To find the average value of a product of two periodic functions with the same period $\tau = 2\pi/\omega$, let us assume two functions f_1 and f_2 given by

$$f_1 = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t} \quad (8)$$

$$f_2 = \sum_{m=-\infty}^{\infty} b_m e^{im\omega t} \quad (9)$$

Then average of the product is given by

$$average\ of\ the\ product = \frac{1}{\tau}\int_0^{\tau} f_1(t)\, f_2(t)\, dt$$

$$= \frac{1}{\tau}\int_0^{\tau}\left[\sum_{n=-\infty}^{\infty} a_n e^{in\omega t} \sum_{m=-\infty}^{\infty} b_m e^{im\omega t}\right] dt$$

$$= \frac{1}{\tau}\int_0^{\tau}\left[\sum_{n=-\infty}^{\infty} a_n\, b_m\, e^{i(n+m)\omega t}\right] dt$$

$$= \frac{1}{\tau}\sum_{n=-\infty}^{\infty} a_n\, b_m \int_0^{\tau} e^{i(n+m)\omega t}\, dt$$

$$= \frac{1}{\tau}\sum_{n=-\infty}^{\infty} a_n\, b_{-n} \qquad (10)$$

(3) Transverse vibrations of a string:

Consider the transverse vibrations of a stretched string at the ends. Suppose that the string is initially distorted into some given curve and then allowed to swing. Let the length of the string be l and the equation of the curve be $y = f(x)$ with respect to the position of equilibrium of the string as $X - axis$ and the one of the ends as origin. The vibrations are given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

The boundary conditions are:

$$y = 0\ at\ x = 0 \qquad (2)$$

$$y = 0\ at\ x = l \qquad (3)$$

$$y = f(x)\ when\ t = 0 \qquad (4)$$

$$\frac{\partial y}{\partial t} = 0\ when\ t = 0 \qquad (5)$$

Let

$$y = Ae^{\alpha x + \beta t} \qquad (6)$$

be the solution of equation (1).

Its first and second derivative with respect to x and t are:

$$\frac{\partial y}{\partial x} = \alpha\left[Ae^{\alpha x+\beta}\ \right] = \alpha y, \qquad \frac{\partial^2 y}{\partial x^2} = \alpha^2 y, \qquad \frac{\partial y}{\partial t} = \beta y, \qquad \frac{\partial^2 y}{\partial t^2} = \beta^2 y \quad (7)$$

Substituting these values in equation (1), we get

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \qquad \beta^2 y = \alpha^2 a^2 y, \qquad \therefore \beta^2 = \alpha^2 a^2$$

$$\beta = \pm \alpha a \qquad (8)$$

Equation (6) becomes

$$y = Ae^{\alpha x + \beta} \; = Ae^{\alpha x \pm \alpha a t} \qquad (9)$$

Equation (9) is the solution of equation (1).

Now substitute $\alpha = \alpha i$ and $\alpha = -\alpha i$ in equation (9), we get

$$y = Ae^{(x \pm at)\alpha i} \qquad (10)$$

$$y = Ae^{-(x \pm at)\alpha i} \qquad (11)$$

By adding equation (10) and (11), we get

$$2y = Ae^{(x \pm at)\alpha i} + Ae^{-(x \pm at)\alpha i} = A\left[e^{(x \pm at)\alpha i} + e^{-(x \pm at)\alpha i}\right]$$

$$y = A\left[\frac{e^{(x \pm at)\alpha i} + e^{-(x \pm at)\alpha i}}{2}\right] = A\cos\alpha(x \pm at) \qquad (12)$$

This may be also expressed as

$$y = B\sin\alpha(x \pm at) \qquad (13)$$

From equation (12), we can write as

$$y = A\cos\alpha(x + at) = A[\cos\alpha x \cos\alpha a t - \sin\alpha x \sin\alpha a t] \qquad (14)$$

$$y = A\cos\alpha(x - at) = A[\cos\alpha x \cos\alpha a t + \sin\alpha x \sin\alpha a t] \qquad (15)$$

Successive solutions of these two will be

$$y = A\cos\alpha x \cos\alpha a t \qquad (16)$$

$$y = A\sin\alpha x \sin\alpha a t \qquad (17)$$

Similarly, from equation (13), we get

$$y = B\sin\alpha x \cos\alpha a t \qquad (18)$$

$$y = B\cos\alpha x \sin\alpha a t \qquad (19)$$

Out of these four values of y, if we take value of y as in equation (18), we get

$$y = B\sin\alpha x \cos\alpha a t \qquad (18)$$

Which satisfied boundary condition shown in equation (2) and (4), and also satisfy condition shown in equation (3) by putting $\alpha = \frac{n\pi}{l}$, equation (18) now becomes

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \qquad (20)$$

By expanding the summation,

$$y = b_1 \sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + b_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + b_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \cdots \qquad (21)$$

This relation satisfied condition (2), (3) and (5). This may also satisfy condition (4) if we put $t = 0$, we get

$$y = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \cdots \qquad (22)$$

Now consider the Fourier series defined by (same as equation (22))

$$f(x) = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \cdots \qquad (23)$$

Then, (multiply both sides of equation (22) with $\sin\frac{n\pi x}{l}$ and integrating from $x = 0$ to $x = l$.)

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad (24)$$

$$b_n = \frac{2}{l}\int_0^l f(\vartheta)\sin\frac{n\pi\vartheta}{l}\,d\vartheta \qquad (25)$$

(By replacing x with ϑ)

Substitute this value of b_n in equation (20), we get

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(\vartheta)\sin\frac{n\pi\vartheta}{l}\,d\vartheta\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

By rearranging the terms, we get

$$y = \frac{2}{l}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\int_0^l f(\vartheta)\sin\frac{n\pi\vartheta}{l}\,d\vartheta \qquad (26)$$

This is a required expression.

Diffusion equation or Fourier equation of heat flow:

Assuming that the temperature at any point (x, y, z) of a solid at time t is $u(x, y, z, t)$, the thermal conductivity of the solid is K, the density of the solid is ρ, specific heat is σ, the heat equation is

$$\frac{\partial u}{\partial t} = h^2 \, \nabla^2 u \quad (1)$$

Equation (1) is called the diffusion equation or Fourier equation of heat flow. Here

$$h^2 = \frac{K}{\rho\sigma} = k \quad (2)$$

is known as diffusivity. We know that heat flows from points at higher temperature to the points at lower temperature and the rate of decrease of temperature at any point varies with the direction. In other words, the amount of heat says ΔH crossing an element of surface ΔS in Δt seconds is proportional to the greatest rate of decrease of the temperature u, i.e.

$$\Delta H = K \, \Delta S \, \Delta t \left|\frac{\partial u}{\partial t}\right| \qquad (3)$$

If $\vec{v}$ be the velocity of heat flow given by

$$\vec{v} = -K \, grad \, u = -K \, \vec{\nabla} u \quad (4)$$

Here $u(x, y, z, t)$ is the temperature of the solid at (x, y, z) at an instant of time 't' and 'K' the thermal conductivity of the solid and its unit is $\frac{cal}{cm \cdot sec \cdot {}^\circ C}$.

Let S be the surface of an arbitrary volume V of the solid. Then the total flux of heat flow across S per unit time is given by

$$H = \iint_S \vec{v} \cdot \overrightarrow{ds}$$

$$H = \iint_S \left(-K \, \vec{\nabla} u\right) \cdot \hat{n} \, ds \qquad (5)$$

Here $\hat{n}$ is the vector normal to the element ds.

We have Gauss's divergence theorem for any vector **A**

$$\iiint_V (\boldsymbol{\nabla} \cdot \mathbf{A}) \, dV = \iint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, dS = \oint\!\!\oint_S \mathbf{A} \cdot d\mathbf{S}$$

Now applying Gauss's divergence theorem according to which if V be the volume bounded by a closed surface S. We have the quantity of heat entering S per unit time as

$$\iint_S \left(K \, \vec{\nabla} u\right) \cdot \hat{n} \, ds = \iiint_V \vec{\nabla} \cdot \left(K \, \vec{\nabla} u\right) dV \qquad (6)$$

Taking volume element $dV = dx\, dv\, dz$,

$$\text{the heat contained in volume } V = \iiint_V \sigma\rho u\, dV \qquad (7)$$

The time rate of increase of heat is given by

$$\frac{\partial}{\partial t}\iiint_V \sigma\rho u\, dV = \iiint_V \sigma\rho \frac{\partial u}{\partial t} dV \qquad (8)$$

Equating R.H.S. of equation (6) and (8), we find

$$\iiint_V \sigma\rho \frac{\partial u}{\partial t} dV = \iiint_V \vec{\nabla}\cdot\left(K\,\vec{\nabla}u\right) dV$$

$$\iiint_V \left[\sigma\rho \frac{\partial u}{\partial t} - \vec{\nabla}\cdot\left(K\,\vec{\nabla}u\right)\right] dV = 0 \qquad (9)$$

But V being arbitrary hence $dV \neq 0$

$$\therefore\ \sigma\rho \frac{\partial u}{\partial t} - \vec{\nabla}\cdot\left(K\,\vec{\nabla}u\right) = 0$$

$$\therefore\ \sigma\rho \frac{\partial u}{\partial t} = \vec{\nabla}\cdot\left(K\,\vec{\nabla}u\right)$$

$$\therefore\ \frac{\partial u}{\partial t} = \frac{K}{\sigma\rho}\,\vec{\nabla}\cdot\left(\vec{\nabla}u\right) = \frac{K}{\sigma\rho}\nabla^2 u = h^2\nabla^2 u = k\nabla^2 u \qquad (10)$$

Where $h^2 = k = \frac{K}{\sigma\rho}$

Equation (10) is also written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad (11)$$

This is three-dimensional diffusion equation.

One dimensional diffusion equation:

Prove that $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2} = k\frac{\partial^2 u}{\partial x^2}$.

Consider one dimensional flow of electricity in a long-insulated cable and specify the current i and voltage E at any point in the cable by x −coordinate and time variable t.

The potential drop E in a line element δx of length at any point x is given by

$$-\delta E = iR\delta x + L\delta x \frac{\partial i}{\partial t} \qquad (1)$$

Where R and L are respectively resistance and inductance per unit length.

If C and G be respectively capacitance and conductance per unit length, then we have

$$-\delta i = GE\delta x + C\delta x\frac{\partial E}{\partial t} \qquad (2)$$

Divide equation (1) and (2) with δx and by rearranging the terms, we have

$$-\frac{\delta E}{\delta x} = -\frac{\partial E}{\partial x} = iR + L\frac{\partial i}{\partial t}$$

$$\frac{\partial E}{\partial x} + Ri + L\frac{\partial i}{\partial t} = 0 \qquad (3)$$

$$-\frac{\delta i}{\delta x} = -\frac{\partial i}{\partial x} = GE + C\frac{\partial E}{\partial t}$$

$$\frac{\partial i}{\partial x} + GE + C\frac{\partial E}{\partial t} = 0 \qquad (4)$$

Now differentiate equation (3) with respect to x and (4) with respect to t, we have

$$\frac{\partial^2 E}{\partial x^2} + R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x\partial t} = 0 \qquad (5)$$

$$\frac{\partial^2 i}{\partial x\partial t} + G\frac{\partial E}{\partial t} + C\frac{\partial^2 E}{\partial t^2} = 0 \qquad (6)$$

Or

$$\frac{\partial^2 i}{\partial x\partial t} = -G\frac{\partial E}{\partial t} - C\frac{\partial^2 E}{\partial t^2} \qquad (7)$$

Substitute value of $\frac{\partial^2 i}{\partial x\partial t}$ from equation (7) into equation (5), we have

$$\frac{\partial^2 E}{\partial x^2} + R\frac{\partial i}{\partial x} + L\left[-G\frac{\partial E}{\partial t} - C\frac{\partial^2 E}{\partial t^2}\right] = 0$$

$$\frac{\partial^2 E}{\partial x^2} + R\frac{\partial i}{\partial x} - GL\frac{\partial E}{\partial t} - CL\frac{\partial^2 E}{\partial t^2} = 0$$

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + GL\frac{\partial E}{\partial t} - R\frac{\partial i}{\partial x} \qquad (8)$$

But from equation (4),

$$\frac{\partial i}{\partial x} = -GE - C\frac{\partial E}{\partial t}$$

Therefore equation (8) becomes

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + GL\frac{\partial E}{\partial t} - R\left[-GE - C\frac{\partial E}{\partial t}\right]$$

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + GL\frac{\partial E}{\partial t} + RGE + RC\frac{\partial E}{\partial t}$$

By rearranging the terms

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + (CR + GL)\frac{\partial E}{\partial t} + RGE \quad (9)$$

Now differentiate equation (3) with respect to t and (4) with respect to x, we have

$$\frac{\partial^2 E}{\partial x \partial t} + R\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} = 0 \quad (10)$$

Or

$$\frac{\partial^2 E}{\partial x \partial t} = -R\frac{\partial i}{\partial t} - L\frac{\partial^2 i}{\partial t^2} \quad (11)$$

$$\frac{\partial^2 i}{\partial x^2} + G\frac{\partial E}{\partial x} + C\frac{\partial^2 E}{\partial x \partial t} = 0 \quad (12)$$

Substitute value of $\frac{\partial^2 E}{\partial x \partial t}$ from equation (11) into equation (12), we have

$$\frac{\partial^2 i}{\partial x^2} + G\frac{\partial E}{\partial x} + C\left[-R\frac{\partial i}{\partial t} - L\frac{\partial^2 i}{\partial t^2}\right] = 0$$

$$\frac{\partial^2 i}{\partial x^2} + G\frac{\partial E}{\partial x} - CR\frac{\partial i}{\partial t} - CL\frac{\partial^2 i}{\partial t^2} = 0$$

$$\frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + CR\frac{\partial i}{\partial t} - G\frac{\partial E}{\partial x} \quad (13)$$

But from equation (3),

$$\frac{\partial E}{\partial x} = -Ri - L\frac{\partial i}{\partial t}$$

Therefore equation (13) becomes

$$\frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + CR\frac{\partial i}{\partial t} - G\left[-Ri - L\frac{\partial i}{\partial t}\right]$$

$$\frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + CR\frac{\partial i}{\partial t} + GRi + GL\frac{\partial i}{\partial t}$$

By rearranging the terms

$$\frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + (CR + GL)\frac{\partial i}{\partial t} + GRi \quad (14)$$

By rewriting equation (9) and (14),

$$\frac{\partial^2 E}{\partial x^2} = CL\frac{\partial^2 E}{\partial t^2} + (CR + GL)\frac{\partial E}{\partial t} + RGE \quad (9)$$

$$\frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + (CR + GL)\frac{\partial i}{\partial t} + GRi \quad (14)$$

Equation (9) and (14) follows that E and i satisfy a second order partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} = CL\frac{\partial^2 u}{\partial t^2} + (CR + GL)\frac{\partial u}{\partial t} + GRu \quad (15)$$

Which is known as ***telegraphy equation***.

If the leakage to the ground is small then $G = L = 0$ and hence equation (15) reduces

$$\frac{\partial^2 u}{\partial x^2} = CR\frac{\partial u}{\partial t} = \frac{1}{k}\frac{\partial u}{\partial t} \quad (16)$$

Here $k = \frac{1}{CR}$.

Equation (16) is called **one dimensional diffusion equation**.

Both the ends of a bar at temperature zero:

If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $F(x)$ in the bar, then find the temperature at a subsequent time t.

Proof: One dimensional heat equation is

$$\frac{\partial u}{\partial t} = h^2\frac{\partial^2 u}{\partial x^2} \quad (1)$$

We have to find a function $u(x,t)$ satisfying equation (1) with the boundary conditions for $t \gg 0$

$$u(x,t) = u(0,t) = 0 \ \ at\ x = 0 \quad (2)$$

$$u(x,t) = u(l,t) = 0 \ \ at\ x = l \quad (3)$$

Here 'l' being the length of bar, for $0 < x < l$

$$u(x,t) = u(x,0) = F(x) \ \ at\ t = 0 \quad (4)$$

In order to apply the method of separation of variables, let us assume that

$$u(x,t) = X(x)\,T(t) \quad (5)$$

X and T being the function of x and t alone, so that

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt}, \qquad \frac{\partial^2 u}{\partial x^2} = T\,\frac{d^2X}{dx^2} \qquad (6)$$

Their substitution in equation (1) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{h^2T}\frac{dT}{dt} \qquad (7)$$

The L.H.S. of equation (7) is space dependant and R. H. S. of equation (7) is time dependent, hence both sides are constant equal to some constant $-\lambda^2$ (say). Therefore equation (7) becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2 \qquad (8)$$

By comparing L.H.S. of equation (8) with $-\lambda^2$, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\therefore\ \frac{d^2X}{dx^2} + \lambda^2X = 0 \qquad (9)$$

And the general solution of equation (9) is given as

$$X = A\cos\lambda x + B\sin\lambda x \qquad (10)$$

By comparing R.H.S. of equation (8) with $-\lambda^2$, we have

$$\frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2$$

$$\therefore\ \frac{dT}{dt} + \lambda^2h^2T = 0 \qquad (11)$$

The general solution of equation (11) is given as

$$T = C\,e^{-\lambda^2h^2t} \qquad (12)$$

By using boundary condition of equation (2): $u(x,t) = u(0,t) = 0\ \ at\ x = 0$

As $u = 0$, $XT = 0$, $\therefore X = 0$, therefore equation (10) becomes

$$X = A\cos\lambda x + B\sin\lambda x\ \Rightarrow 0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$$

$$\boldsymbol{A = 0} \qquad (13)$$

Therefore equation (10) now becomes

$$X = B\sin\lambda x \qquad (14)$$

By using boundary condition of equation (3): $at\ x = l\ \ u(x,t) = u(l,t) = 0,\ t \gg 0$

Equation (14) becomes,

$$X = B \sin \lambda x \quad \Rightarrow 0 = B \sin \lambda l$$

But $B \neq 0 \quad \therefore \ \sin \lambda l = 0 \ \therefore \lambda l = n\pi$

$$\therefore \lambda = \frac{n\pi}{l}, n = 0, 1, 2, \dots \quad (15)$$

Therefore, solution of equation (1), $u = X\,T$ takes form as:

$$u = X\,T = B \sin \frac{n\pi x}{l} C\, e^{-\frac{n^2\pi^2 h^2 t}{l^2}}$$

And by including C into B,

$$u = B \sin \frac{n\pi x}{l}\ e^{-\frac{n^2\pi^2 h^2 t}{l^2}} \qquad (16)$$

Summing over all values of n, equation (16) becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\ e^{-\frac{n^2\pi^2 h^2 t}{l^2}} \qquad (17)$$

Applying condition of equation (4), $at\ t = 0,\ u(x,t) = u(x,0) = F(x),\ \ 0 < x < l$

We have

$$F(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\ for\ 0 < x < l \qquad (18)$$

To find out B_n, multiply both sides of equation (18) with $\sin \frac{n\pi x}{l}$ and integrating the result from $x = 0\ to\ x = l$, we have

$$B_n = \frac{2}{l}\int_0^l F(x) \sin \frac{n\pi x}{l}\ dx = \frac{2}{l}\int_0^l F(u) \sin \frac{n\pi u}{l}\ du \qquad (19)$$

Hence the required solution by substituting value of equation (19) in equation (17) is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l F(u) \sin \frac{n\pi u}{l}\ du\right] \sin \frac{n\pi x}{l}\ e^{-\frac{n^2\pi^2 h^2 t}{l^2}}$$

By rearranging the terms,

$$\therefore\ u(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 h^2 t}{l^2}} \sin \frac{n\pi x}{l} \int_0^l F(u) \sin \frac{n\pi u}{l}\ du \qquad (20)$$

Which is the required solution.

Two-dimensional diffusion equation:

Consider a thin rectangular plate whose surface is impervious (i.e. not permitting penetration or passage) to heat flow and which has an arbitrary function of temperature $F(x, y)\ at\ t = 0$, its four edges $x = 0, x = a, y = 0, y = b$ are kept at zero temperature. We have to determine the subsequent temperature at a point of the plate as t increases.

Two-dimensional heat equation is written as:

$$\frac{\partial u}{\partial t} = h^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \qquad (for\ all\ t) \quad (1)$$

The boundary conditions are:

$$at\ x = 0 \quad u(x, y, t) = u(0, y, t) = 0 \qquad (2)$$

$$at\ x = a \quad u(x, y, t) = u(a, y, t) = 0 \qquad (3)$$

$$at\ y = 0 \quad u(x, y, t) = u(x, 0, t) = 0 \qquad (4)$$

$$at\ y = b \quad u(x, y, t) = u(x, b, t) = 0 \qquad (5)$$

$$at\ t = 0, \qquad u(x, y, t) = u(x, y, 0) = F(x, y) \quad (6)$$

In order to apply the method of separation of variables, let us assume that

$$u(x, y, t) = X(x)\, Y(y)\, T(t) \qquad (7)$$

Here X is a function of x alone, Y is a function of y alone and T is a function of t alone, so that

$$\frac{\partial u}{\partial t} = XY\frac{dT}{dt}, \qquad \frac{\partial^2 u}{\partial x^2} = YT\,\frac{d^2X}{dx^2}, \qquad \frac{\partial^2 u}{\partial y^2} = XT\,\frac{d^2Y}{dy^2} \quad (8)$$

Substituting values of equation (8) in equation (1), we have

$$\frac{\partial u}{\partial t} = h^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \Rightarrow XY\frac{dT}{dt} = h^2\left[YT\,\frac{d^2X}{dx^2} + XT\,\frac{d^2Y}{dy^2}\right]$$

Now divide both sides of above equation with h^2XYT

$$\frac{1}{h^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} \quad (9)$$

In equation (9), L.H.S. is time dependent only and R.H.S. is space dependent only. Their equality suggests that both sides equal to some constant $= -\lambda^2$ (say). We can assume

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda_1^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda_2^2, \qquad \frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2 \quad (10)$$

So that

$$\lambda^2 = \lambda_1^2 + \lambda_2^2 \qquad (11)$$

The general solution of equation (10) are

$$X = A\cos\lambda_1 x + B\sin\lambda_1 x \qquad (12)$$

$$Y = C\cos\lambda_2 y + D\sin\lambda_2 y \qquad (13)$$

$$T = E\,e^{-\lambda^2 h^2 t} \qquad (14)$$

Therefore, solution of equation (1) i.e. equation (7) becomes

$$u(x,y,t) = XYT = (A\cos\lambda_1 x + B\sin\lambda_1 x)(C\cos\lambda_2 y + D\sin\lambda_2 y)E\,e^{-\lambda^2 h^2 t} \qquad (15)$$

By using boundary condition of equation (2): $at\ x = 0 \quad u(x,y,t) = u(0,y,t) = 0$

As $u = 0, \quad XYT = 0, \quad \therefore X = 0$, therefore equation (12) becomes

$$0 = (A\cos 0 + B\sin 0)$$

$$\therefore\ \boldsymbol{A} = \boldsymbol{0} \qquad (16)$$

Therefore equation (12) now becomes

$$X = B\sin\lambda_1 x \qquad (17)$$

By using boundary condition of equation (3): $at\ x = a \quad u(x,y,t) = u(a,y,t) = 0$

Equation (17) becomes,

$$X = B\sin\lambda_1 x \ \Rightarrow 0 = B\sin\lambda_1 a$$

But $B \neq 0 \quad \therefore\ \sin\lambda_1 a = 0 \ \therefore \lambda_1 a = m\pi$

$$\therefore \lambda_1 = \frac{m\pi}{a}, m = 0,1,2,\ldots \qquad (18)$$

Similarly, by using boundary conditions of equation (4) and (5) we obtain

$$C = 0 \qquad (19)$$

$$\lambda_2 = \frac{n\pi}{b}, n = 0,1,2,\ldots \qquad (20)$$

Therefore, equation (15) becomes

$$u(x,y,t) = XYT = (B\sin\lambda_1 x)(D\sin\lambda_2 y)E\,e^{-\lambda^2 h^2 t}$$

As B, D, E are constants, by merging D & E into B, above equation becomes

$$u(x,y,t) = B\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,e^{-\lambda^2 h^2 t} \qquad (21)$$

From equation (11), $\lambda^2 = \lambda_1^2 + \lambda_2^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right] = \lambda_{mn}^2$

$$u(x,y,t) = B\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,e^{-\lambda_{mn}^2 h^2 t} \qquad (22)$$

Summing over all possible values of m *and* n, the general solution: equation (22) becomes

$$u(x,y,t) = \sum_{m,n=1}^{\infty} B_{mn}\,e^{-\lambda_{mn}^2 h^2 t}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (23)$$

Here B_{mn} are arbitrary constants. By applying condition of equation (6),

$$at\ t = 0, \qquad u(x,y,t) = u(x,y,0) = F(x,y)$$

We have

$$F(x,y) = u(x,y,0) = \sum_{m,n=1}^{\infty} B_{mn}\,\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (24)$$

To determine B_{mn} we multiply both sides of equation (24) with $\sin\frac{m\pi x}{a}$ and $\sin\frac{n\pi y}{b}$ and integrating the result from $x = 0\ to\ x = a$ and $y = 0\ to\ y = b$ we have

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b F(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy = \frac{4}{ab}\int_0^a\int_0^b F(u,v)\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\,du\,dv \qquad (25)$$

Hence the complete solution by substituting value of equation (25) in equation (23) is

$$u_{mn}(x,y,t) = \sum_{m,n=1}^{\infty}\left[\int_0^a\int_0^b F(u,v)\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\,du\,dv\right]e^{-\lambda_{mn}^2 h^2 t}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

By rearranging the terms,

$$u_{mn}(x,y,t) = \frac{4}{ab}\sum_{m,n=1}^{\infty} e^{-\lambda_{mn}^2 h^2 t}\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b}\int_0^a\int_0^b F(u,v)\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\,du\,dv \qquad (26)$$

Which is the required equation.

The wave equation:

Derivation of one-dimensional wave equation:

Consider a flexible string of length l tightly stretched between two points $x = 0$ and $x = l$ on $X - axis$. If the string is set into small transverse vibrations, the displacement $u(x,t)$ from the $X - axis$ of any point x of the string at any time t is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Where $c^2 = \frac{T}{\rho}$, where T being tension and ρ the linear density. Equation (1) is called one dimensional wave equation.

Let the string (assumed to be perfectly flexible) of length l tightly stretched between the points $x = 0$ and $x = l$ on $X - axis$ be distorted and then at a certain instant of time say $t = 0$, it is released and allowed to vibrate. To determine its deflection (displacement from the $X - axis$) at any point x at any time t, let us take the following assumptions:

(i) The string is uniform. i.e. its mass m per unit length is constant.
(ii) The string is perfectly elastic so offer no resistance to any bending.
(iii) The tension T is so Large that the action of gravitational force on the string is negligible.
(iv) The motion of the string is a small transverse vibration in a vertical plane.

Consider the motion of an element PQ of length δs of the string. The string being perfectly elastic, hence tension T_1 at P and T_2 at Q are tangential to the curve of the string. Let T_1 and T_2 make angle α and β respectively with the horizontal.

There being no motion in the horizontal direction, we have

$$T_1 \cos\alpha = T_2 \cos\beta = T = constant \quad (2)$$

$$mass\ of\ the\ element\ PQ = \rho\ \delta s \quad (3)$$

By Newton's second law of motion ($F = ma$), we have

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\ \delta s)\frac{\partial^2 u}{\partial t^2} \quad (4)$$

Now divide equation (4) with equation (2), we have

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \left(\frac{\rho\ \delta s}{T}\right)\frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho\ \delta s}{T}\ \frac{\partial^2 u}{\partial t^2} \quad (5)$$

Replacing δs by δx since the gradient of the curve is very small equation (5) becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\ \delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho\ \delta s}{T}\ \frac{\partial^2 u}{\partial t^2} = \frac{\rho\ \delta x}{T}\ \frac{\partial^2 u}{\partial t^2} \quad (6)$$

Since $\tan\alpha$ and $\tan\beta$ are slopes at x and $x +\ \delta x$ respectively.

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\ \delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T}\ \frac{\partial^2 u}{\partial t^2}$$

i.e.

$$\frac{u_x(x +\ \delta x, t) - u_x(x, t)}{\delta x} = \frac{\rho}{T}\ \frac{\partial^2 u}{\partial t^2}$$

By taking the limit $\delta x \to 0$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad (7)$$

Or

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (8)$$

Where $\frac{1}{c^2} = \frac{\rho}{T}$

Which is required expression.

Derivation of two-dimensional wave equation:

Consider a rectangular membrane, for which the two-dimensional wave equation is written as

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad (1)$$

Consider the motion of a stretched membrane supposed to be stretched and fixed along its entire boundary in the $X - Y\ plane$. Let us take the following assumptions.

(i) The membrane is homogeneous. i.e. mass ρ per unit area is constant.
(ii) The membrane is perfectly flexible and so thin that it offers no resistance to any bending.
(iii) The tension T per unit length caused by the stretching of the membrane is invariant during the motion. It retains the same value at each of its points and in all the directions.
(iv) The deflection $u(x, y, t)$ of the membrane during the motion is negligible as compared to the size of the membrane. Also, all the angles of inclination are small.

consider the motion of an element $ABCD$ of the membrane. Let its area be $\delta x \delta y$. The tension per unit length is T, the force acting on the edges are $T\delta x$ and $T\delta y$ approximately. Also, the membrane being perfectly flexible, the tension $T\delta x$ and $T\delta y$ are tangential to the membrane. Let α, β be the inclinations of these tensions with the horizontal. Then the horizontal components of these forces are $T\delta y \cos\alpha$ and $T\delta y \cos\beta$. When α & β are small, $\cos\alpha \to 1$ and $\cos\beta \to 1$ so that $T\delta y\cos\alpha \to T\delta y$ and $T\delta y\cos\beta \to T\delta y$. i.e. the horizontal components of the forces at opposite edges are nearly equal and hence the motion of the particles of the membrane in horizontal direction is negligibly small. We assume that every particle of the membrane moves vertically.

$$\text{The resultant vertical force} = T\delta y\sin\beta - T\delta y\sin\alpha$$

$$= T\delta y(\sin\beta - \sin\alpha)$$

($\because \alpha, \beta$ being small $\sin\alpha = \alpha = \tan\alpha$ and $\sin\beta = \beta = \tan\beta$)

$$\text{The resultant vertical force} = T\delta y(\tan\beta - \tan\alpha)$$

$$= T\delta y[u_x(x+\ \delta x, y_1) - u_x(x, y_2)] \qquad (2)$$

Where u_x is the partial derivative w. r. t. x and y_1, y_2 are the values of y between y and $y + \delta y$.

Similarly, the resultant vertical force acting on the other two edges

$$= T\delta x\left[u_y(x_1, y+\ \delta y) - u_y(x_2, y)\right] \qquad (3)$$

Where u_y is the partial derivative w. r. t. y and x_1, x_2 are the values of x between x and $x + \delta x$.

By Newton's second law of motion, we have Total vertical force on the element

$$(F = ma) = \rho\ \delta x\ \delta y \frac{\partial^2 u}{\partial t^2}$$

$$i.e.\ T\delta y[u_x(x+\ \delta x, y_1) - u_x(x, y_2)] + T\delta x\left[u_y(x_1, y+\ \delta y) - u_y(x_2, y)\right] = \rho\ \delta x\ \delta y \frac{\partial^2 u}{\partial t^2}$$

Here $\frac{\partial^2 u}{\partial t^2}$ is the acceleration of the element. Thus

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x+\ \delta x, y_1) - u_x(x, y_2)}{\delta x}\right] + \frac{T}{\rho}\left[\frac{u_y(x_1, y+\ \delta y) - u_y(x_2, y)}{\delta y}\right]$$

By taking limits $\delta x \to 0$ and $\delta y \to 0$, we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[u_{xx} + u_{yy}\right] = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\ \nabla^2 u \qquad (4)$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

This is two-dimensional wave equation.